The Minimum Labeling Spanning Tree Problem: Heuristic and Metaheuristic Approaches

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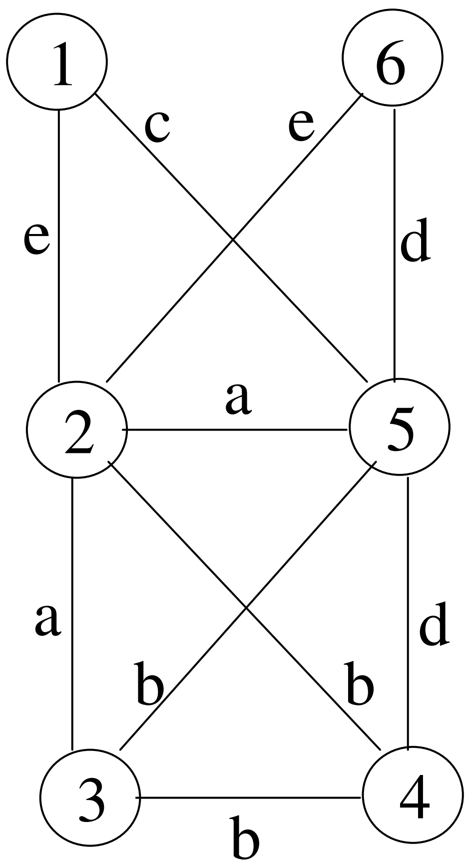
Introduction

- The Minimum Labeling Spanning Tree (MLST) Problem
  - Communications network design
  - Edges may be of different types or media (e.g., fiber optics, cable, microwave, telephone lines, etc.)
  - Each edge type is denoted by a unique letter or color
  - Construct a spanning tree that minimizes the number of colors

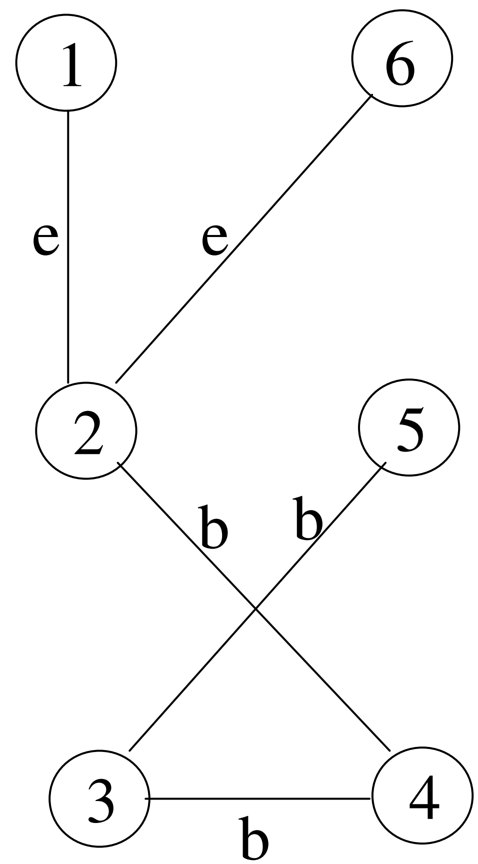
Introduction

- A Small Example

Input



Solution



Literature Review

- Where did we start?
  - The MLST Problem is NP-hard
  - Several heuristics had been proposed
  - One of these, MVCA (version 2), was very fast and effective
  - Worst-case bounds for MVCA had been obtained

Literature Review

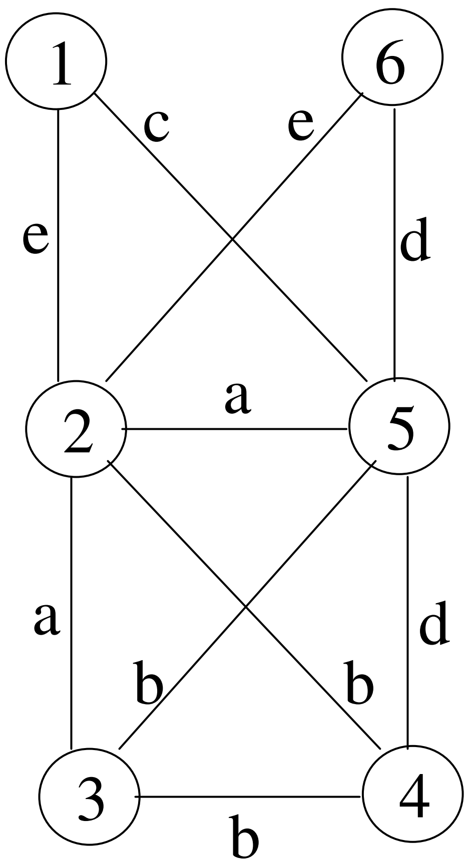
- An optimal algorithm (using backtrack search) had been proposed
- On small problems, MVCA consistently obtained nearly optimal solutions
- See [Chang & Leu, 1997], [Krumke & Wirth, 1998], [Wan, Chen & Xu, 2002], and [Bruggemann, Monnot & Woeginger, 2003]

Description of MVCA

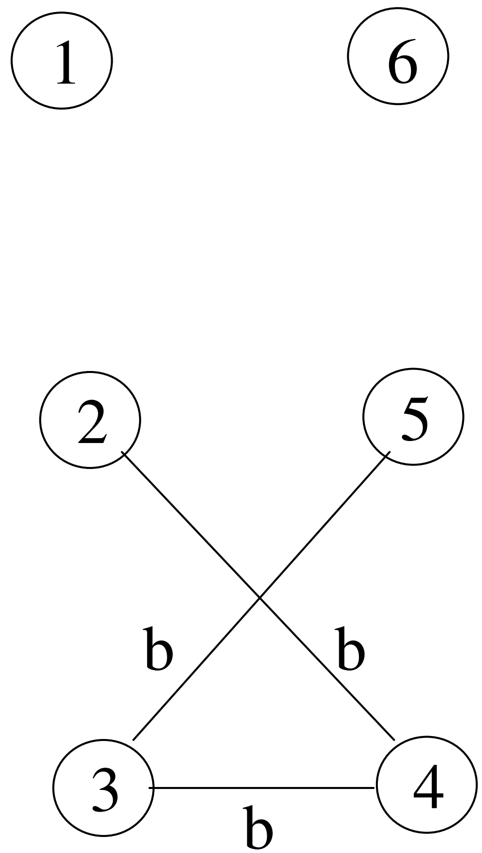
0. Input: G (V, E, L).
1. Let $C \leftarrow \{\ \}$ be the set of used labels.
2. repeat
3. Let H be the subgraph of G restricted to edges with labels from C.
4. for all $i \in L - C$ do
5. Determine the number of connected components when inserting all edges with label i in H.
6. end for
7. Choose label i with the smallest resulting number of components and do: $C \leftarrow C \cup \{i\}$.
8. Until H is connected and spans V.

How MVCA Works

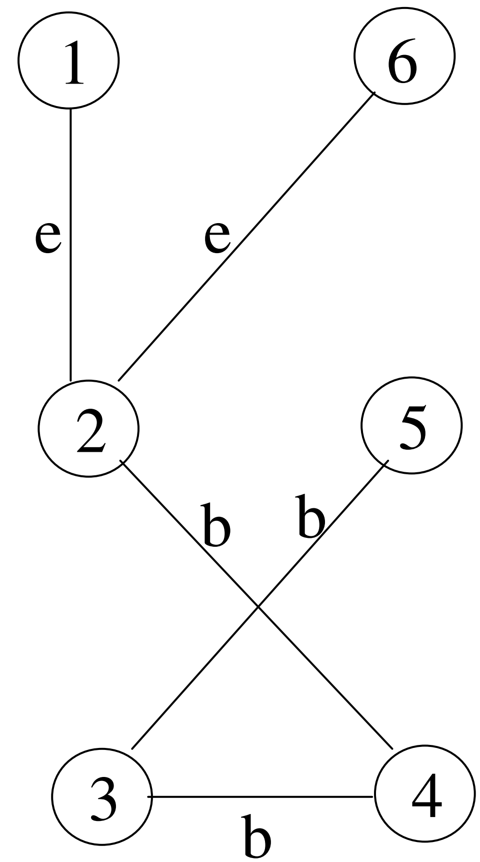
Input



Intermediate Solution



Solution



Worst-Case Results

1. Krumke, Wirth (1998): $$\frac{\text{MVCA}}{\text{OPT}} \le 1 + 2\ln n$$

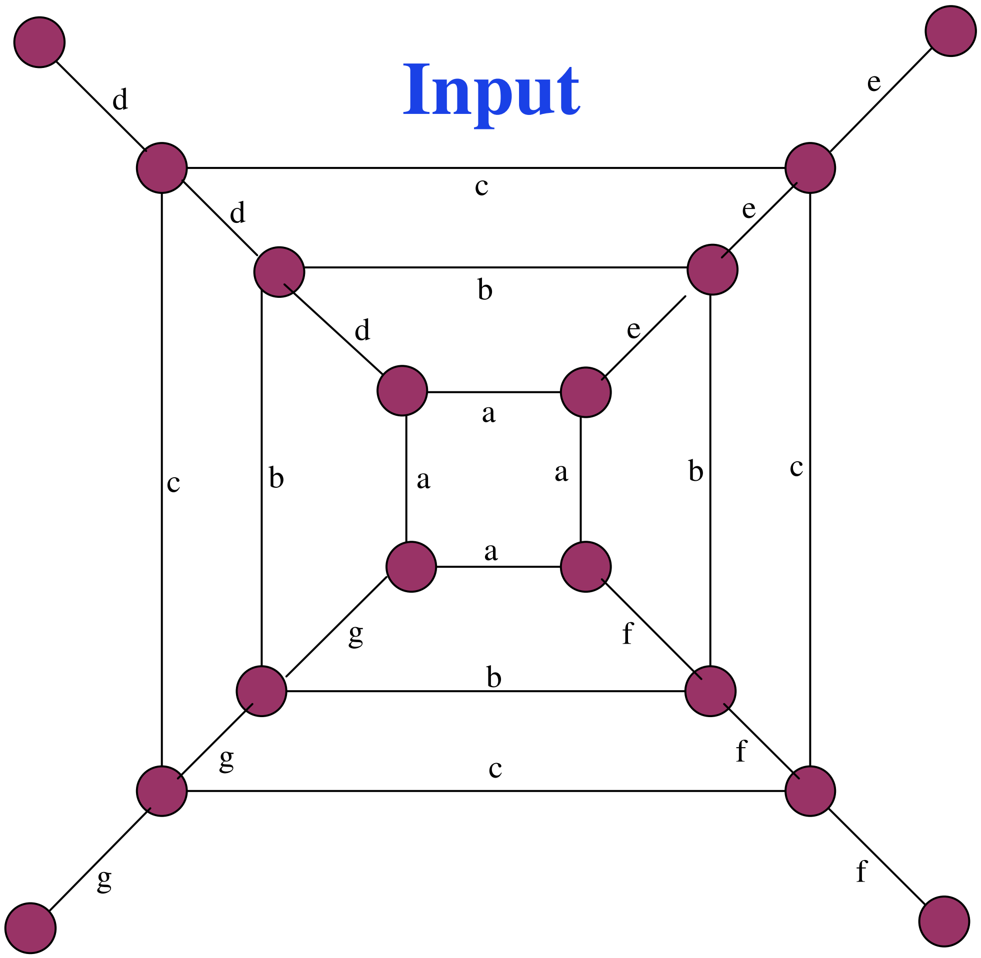
2. Wan, Chen, Xu (2002): $$\frac{\text{MVCA}}{\text{OPT}} \le 1 + \ln(n-1)$$

3. Xiong, Golden, Wasil (2004): $$\frac{\text{MVCA}}{\text{OPT}} \le H_b = \sum_{i=1}^{b} \tfrac{1}{i} < 1 + \ln b$$

where b = max label frequency, and

H_b= b[th] harmonic number

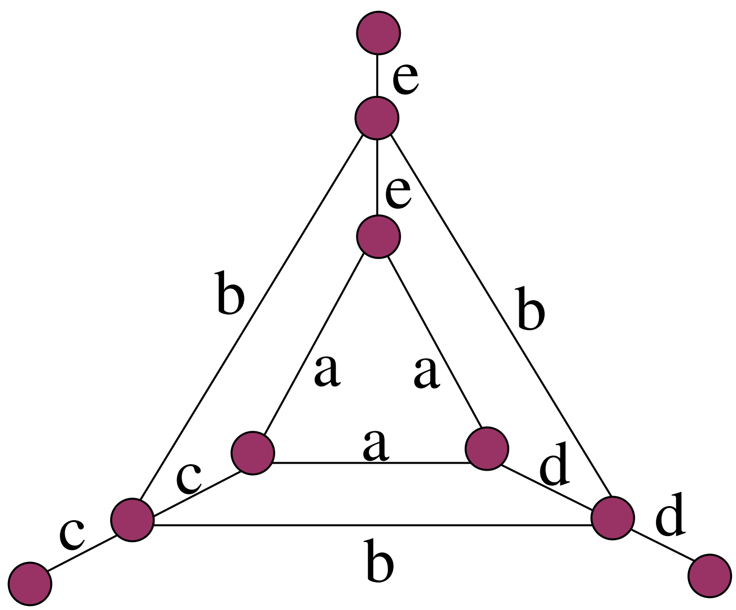
A Perverse-Case Example



- **MVCA**
  - It might add colors associated with 3 polygons first
  - Then, it must add colors associated with 4 rays
- **Optimal Solution**
  - Add colors associated with 4 rays first
  - Then, add color associated with 1 polygon

A Perverse Family of Graphs

- Given an integer k ≥3, build a graph G = (V, E) with k^2 nodes and 2k – 1 labels, where the edges form k –1 concentric polygons and k rays (as below for k = 3)



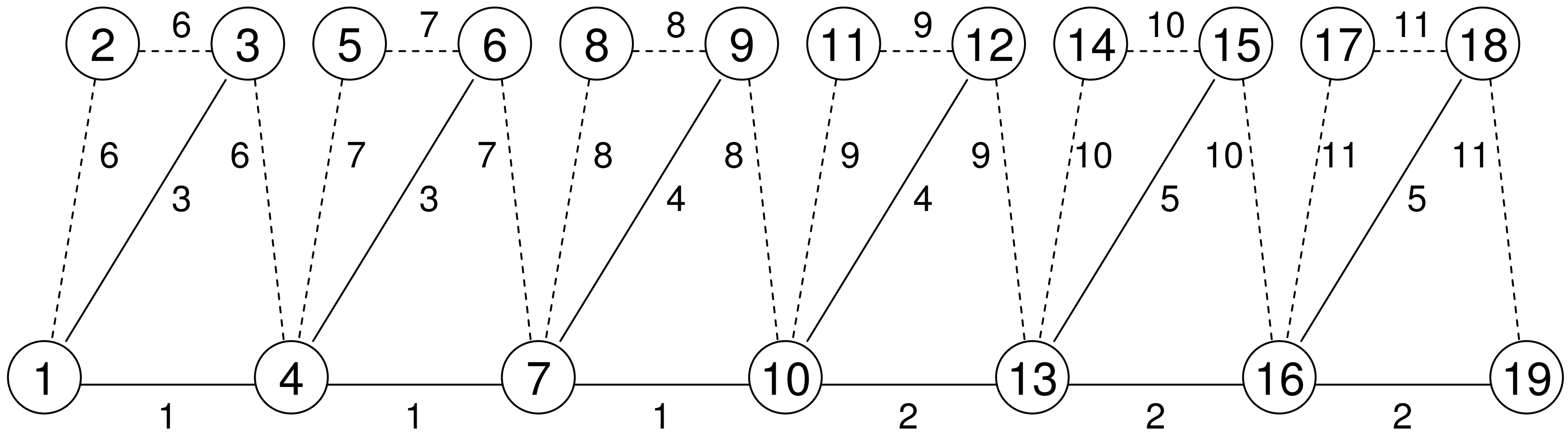
- In the previous example, we had k = 4 and

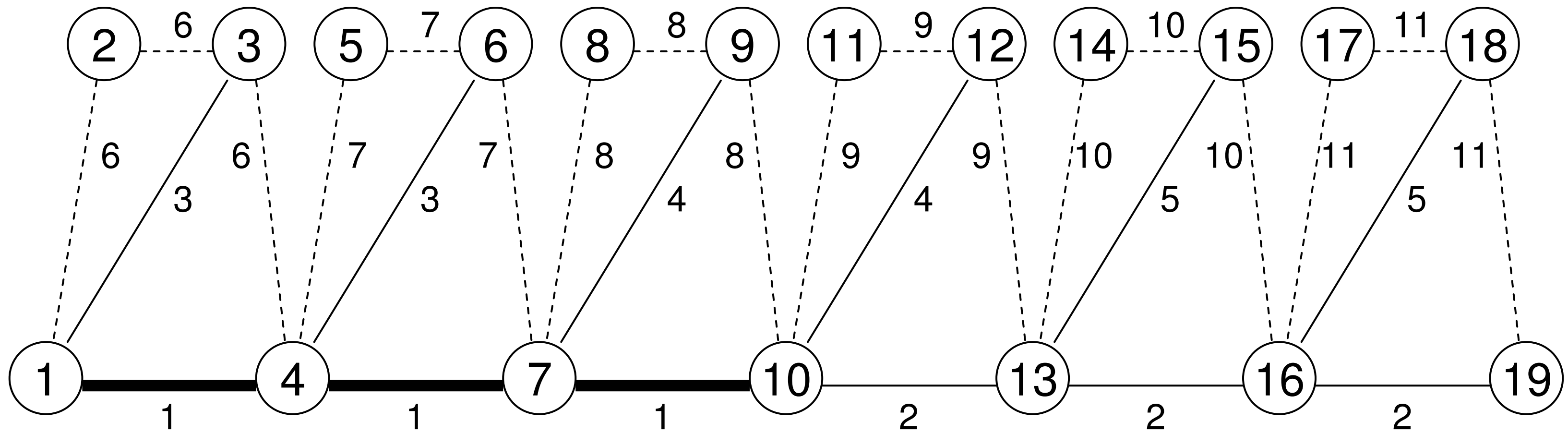
$$\frac{\text{MVCA}}{\text{OPT}} = \frac{7}{5}$$

Some Observations

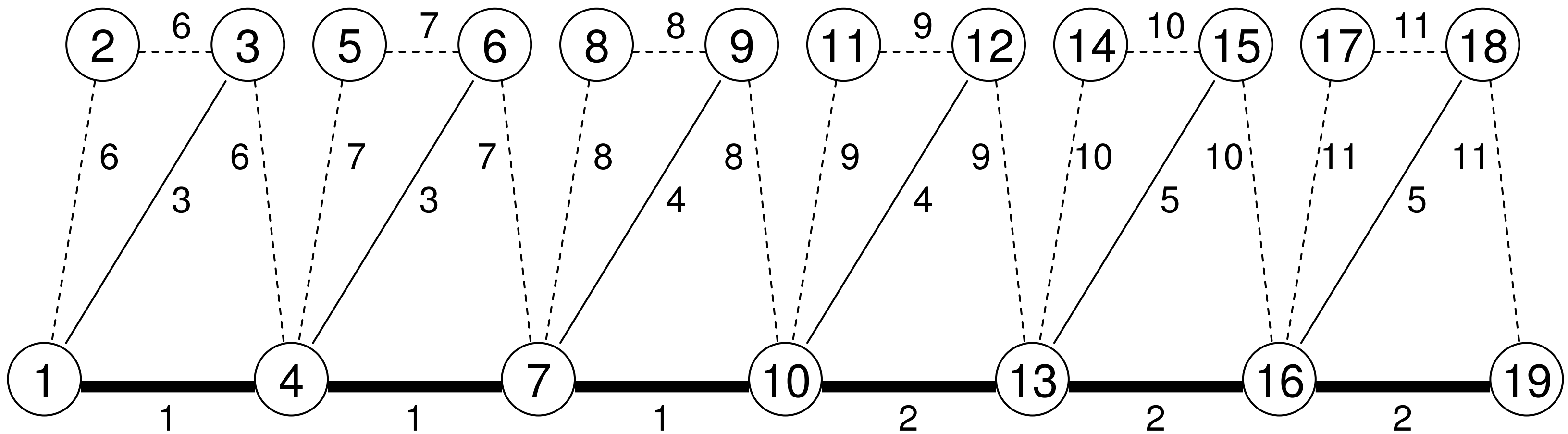
- In general, the result $\frac{\text{MVCA}}{\text{OPT}} = \frac{2k-1}{k+1}$, which approaches 2 for large *k*, is possible
- The labels associated with the rays in the perverse-case example are cut labels
- Definition: A label c is called a cut label if the removal of all edges with label c disconnects graph G
- These labels must be in any solution
- Next, we show that the Xiong, Golden, Wasil worst-case bound is tight

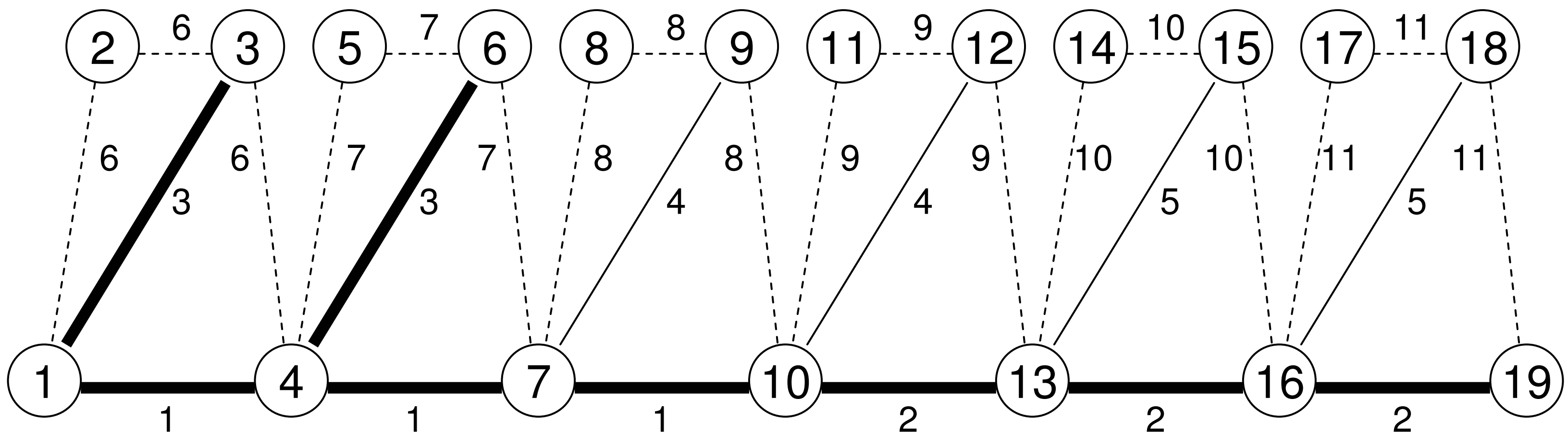
A Worst-Case Example



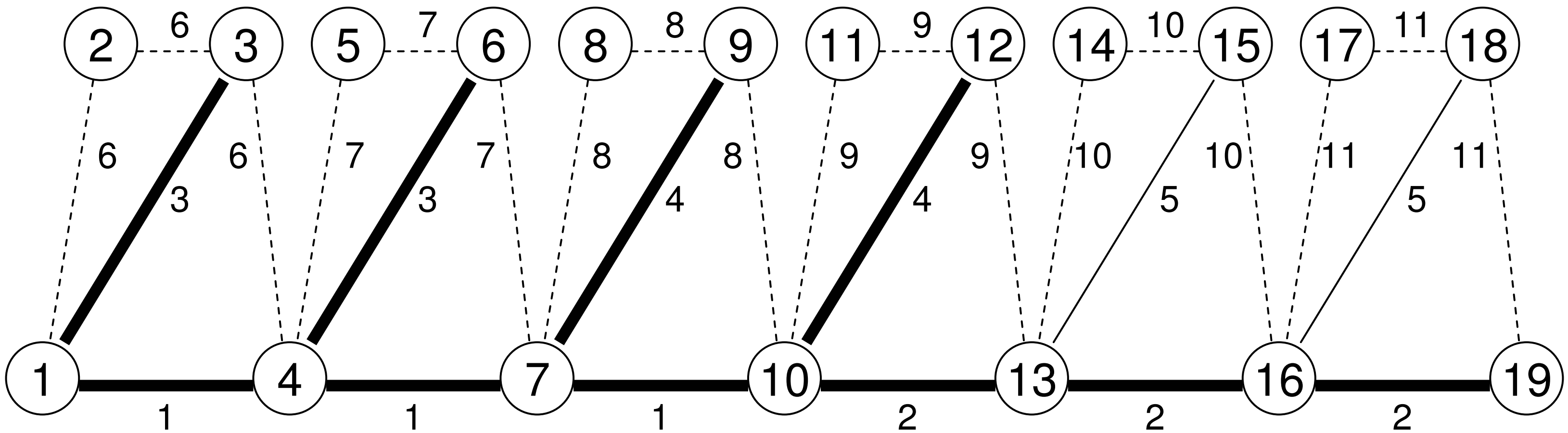


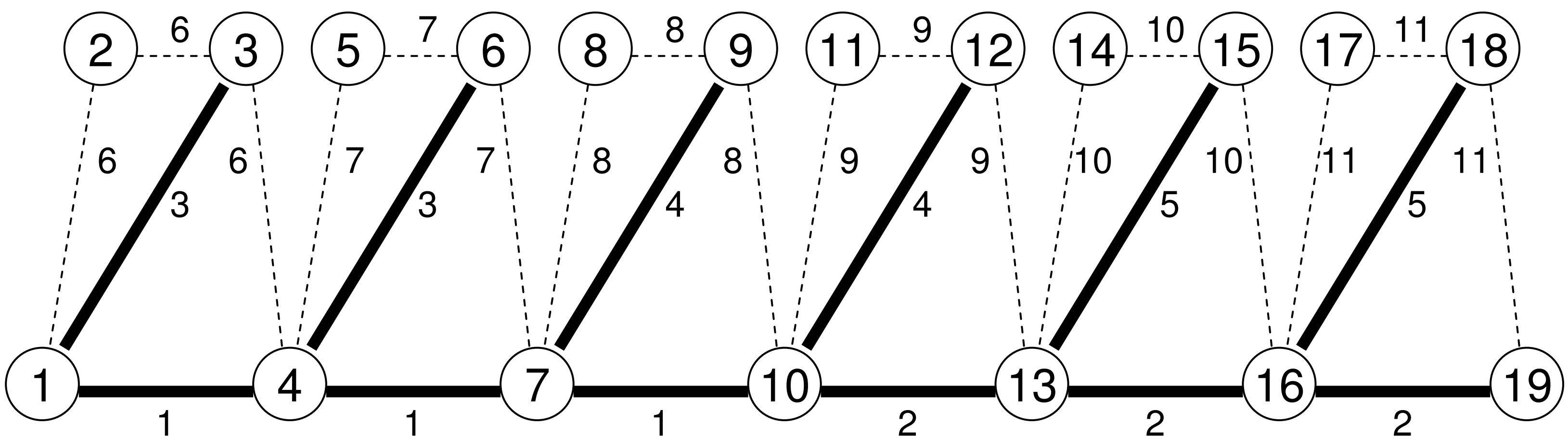
A Worst-Case Example - continued





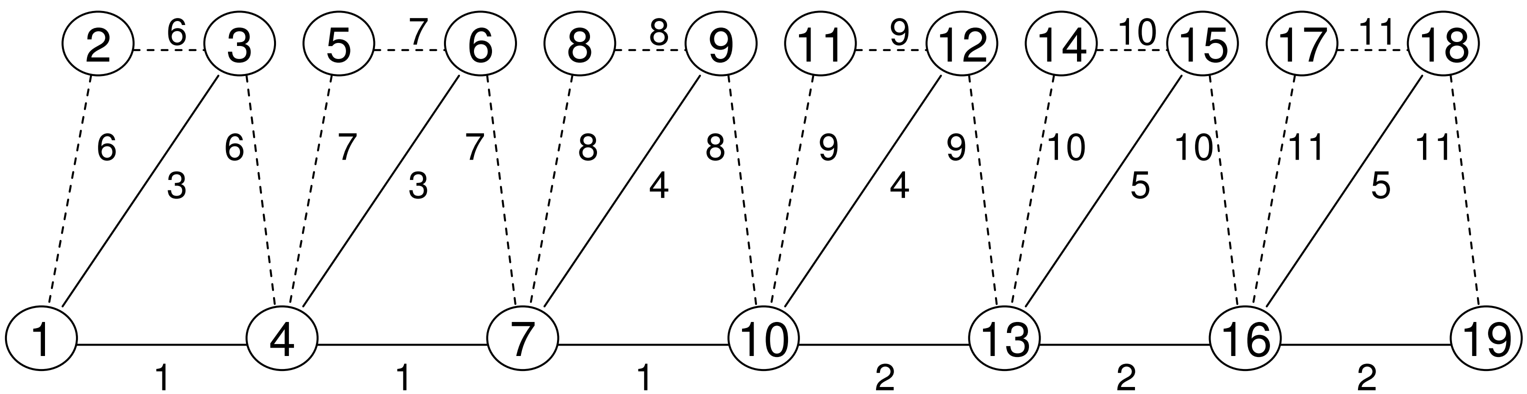
A Worst-Case Example - continued

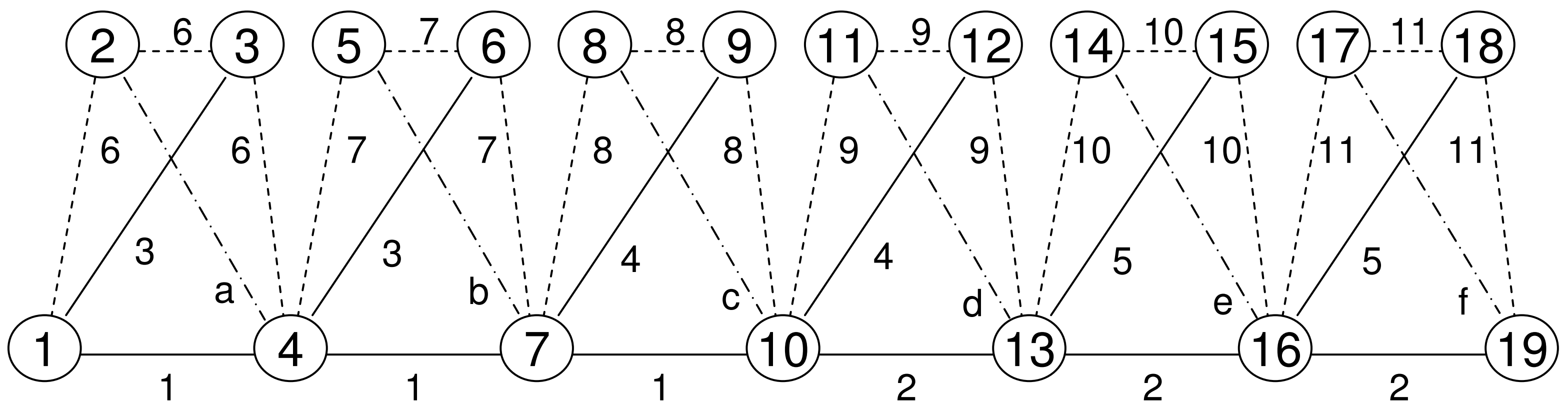




A Worst-Case Example - continued

- Suppose we implement MVCA by adding cut labels first? Is the bound still tight?





More Observations

- For the worst-case example with $b = 3$ and $n = 19$,

$$\frac{\text{MVCA}}{\text{OPT}} = \frac{11}{6} = 1 + \frac{1}{2} + \frac{1}{3} = H_3$$

- Unlike the MST, where we focus on the edges, here it makes sense to focus on the labels or colors

- Next, we present a genetic algorithm (GA) for the MLST problem

Genetic Algorithm: Overview

- Randomly choose p solutions to serve as the initial population
- Suppose s [0], s [1], … , s [p – 1] are the individuals (solutions) in generation 0
- Build generation k from generation k – 1 as below

For each j between 0 and p – 1, do:

t [j] = crossover { s [j], s [(j + k) mod p] }

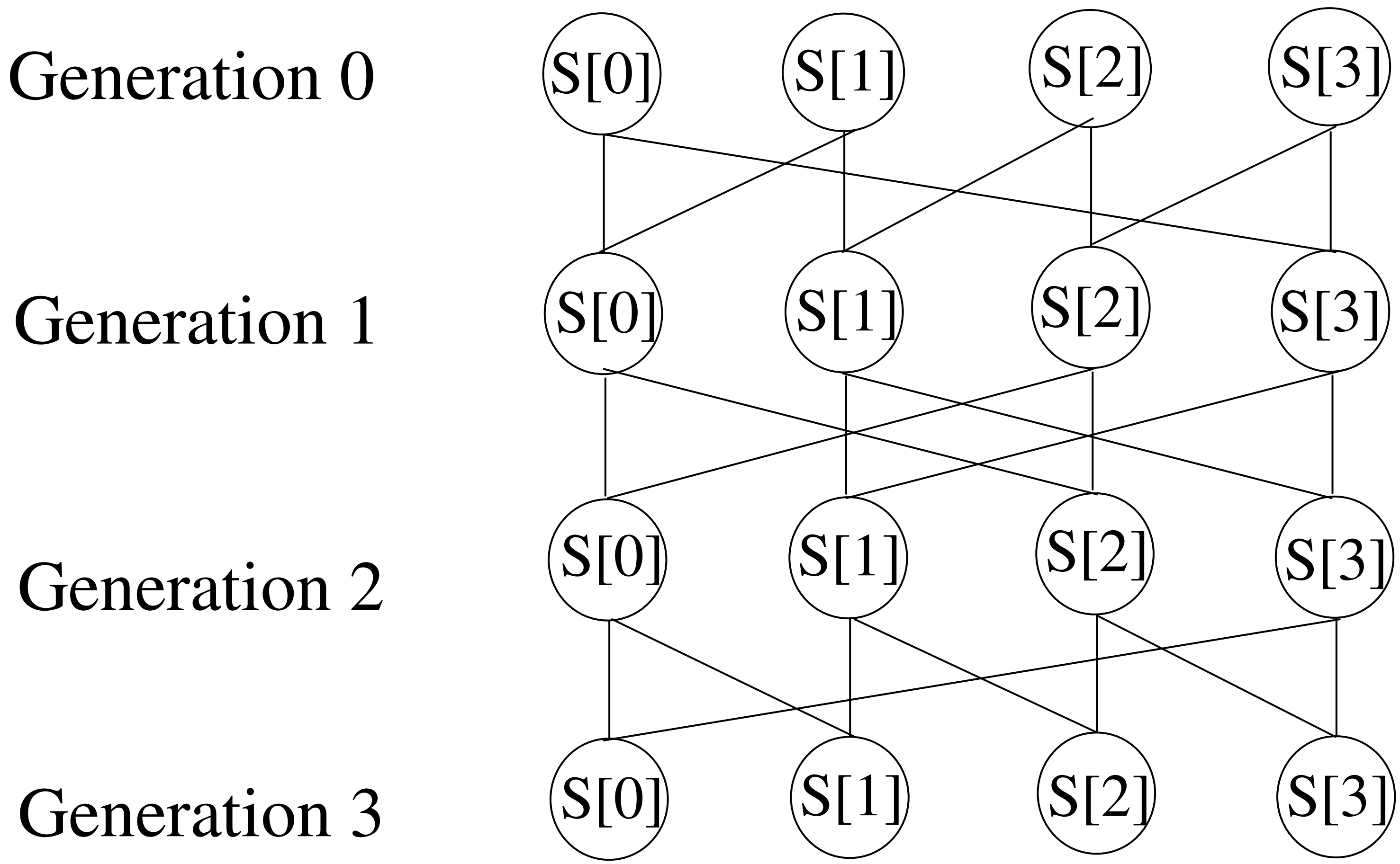
t [j] = mutation { t [j] }

s [j] = the better solution of s [j] and t [j]

End For

- Run until generation p – 1 and output the best solution from the final generation

Crossover Schematic (p = 4)



Crossover

- Given two solutions s [1] and s [2], find the child T = crossover { s [1], s [2] }
- Define each solution by its labels or colors
- Description of Crossover

   a. Let $S = s[1] \cup s[2]$ and T be the empty set

   b. Sort S in decreasing order of the frequency of its labels

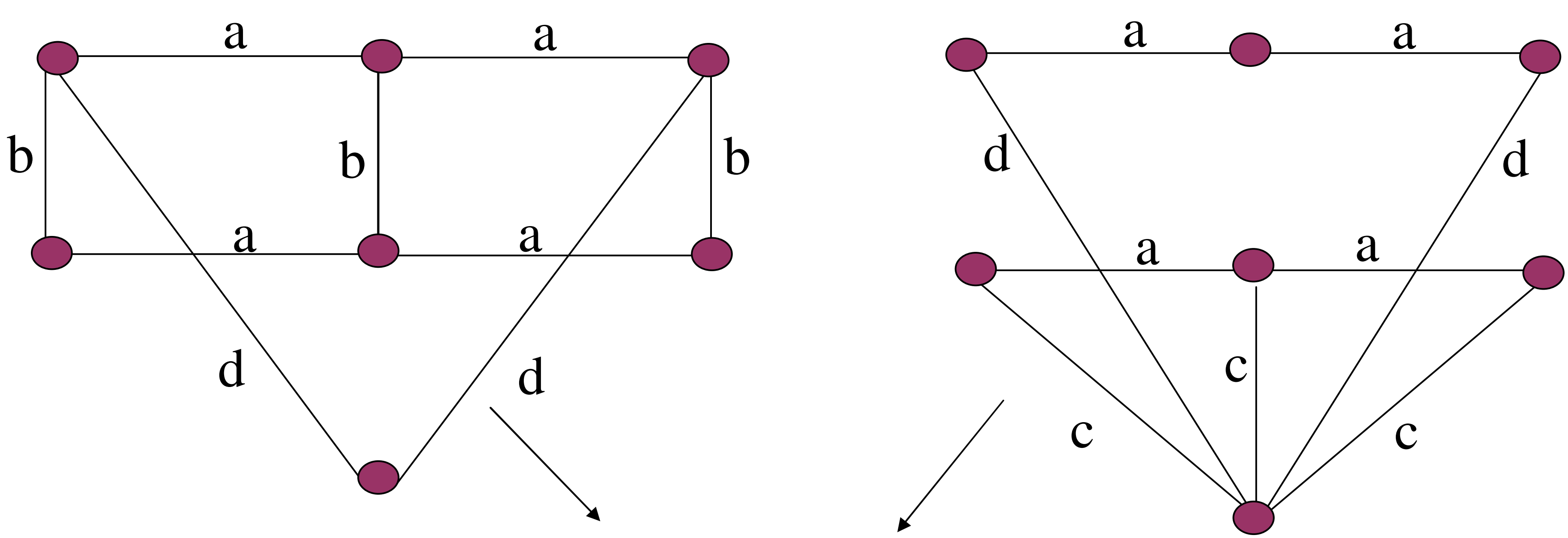
   c. Add labels of S to T, until T represents a connected subgraph H of G that spans V

   d. Output T

An Example of Crossover

s [1] = { a, b, d }

s [2] = { a, c, d }



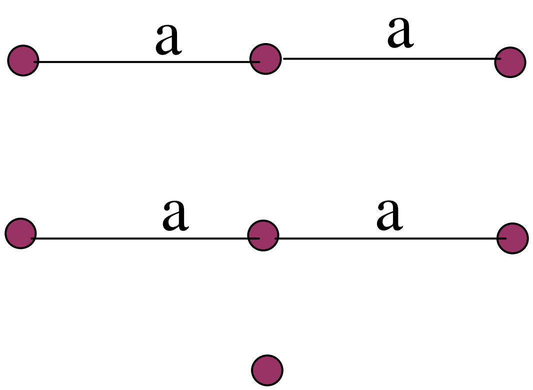
T = {   }

S = { a, b, c, d }

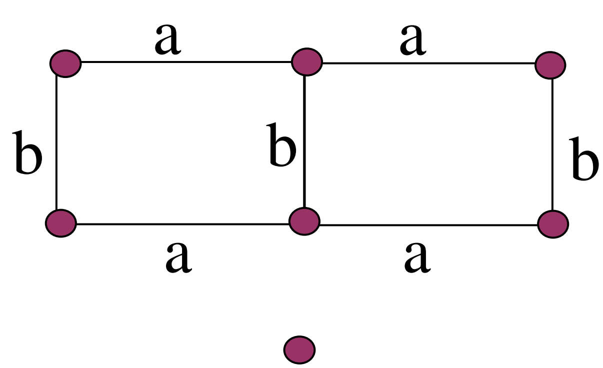
Ordering: a, b, c, d

An Example of Crossover

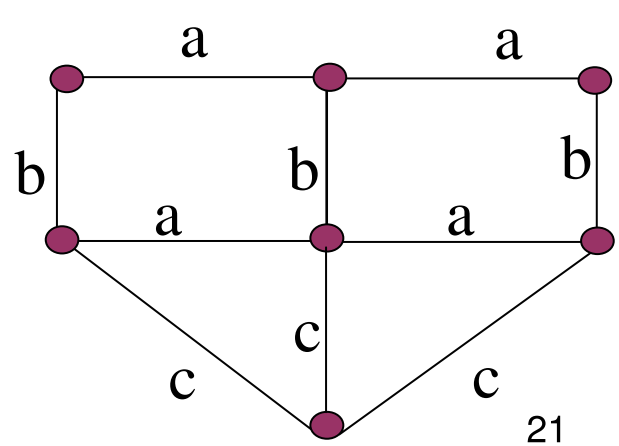
T = { a }



T = { a, b }



T = { a, b, c }



Mutation

- Given a solution S, find a mutation T
- Description of Mutation

  a. Randomly select a label c not in S and let
  $S = S \cup \{ c \}$

  b. Sort S in decreasing order of the frequency of its labels

  c. From the last label on the above list to the first, try to remove one label from S while preserving a connected subgraph H of G that spans V

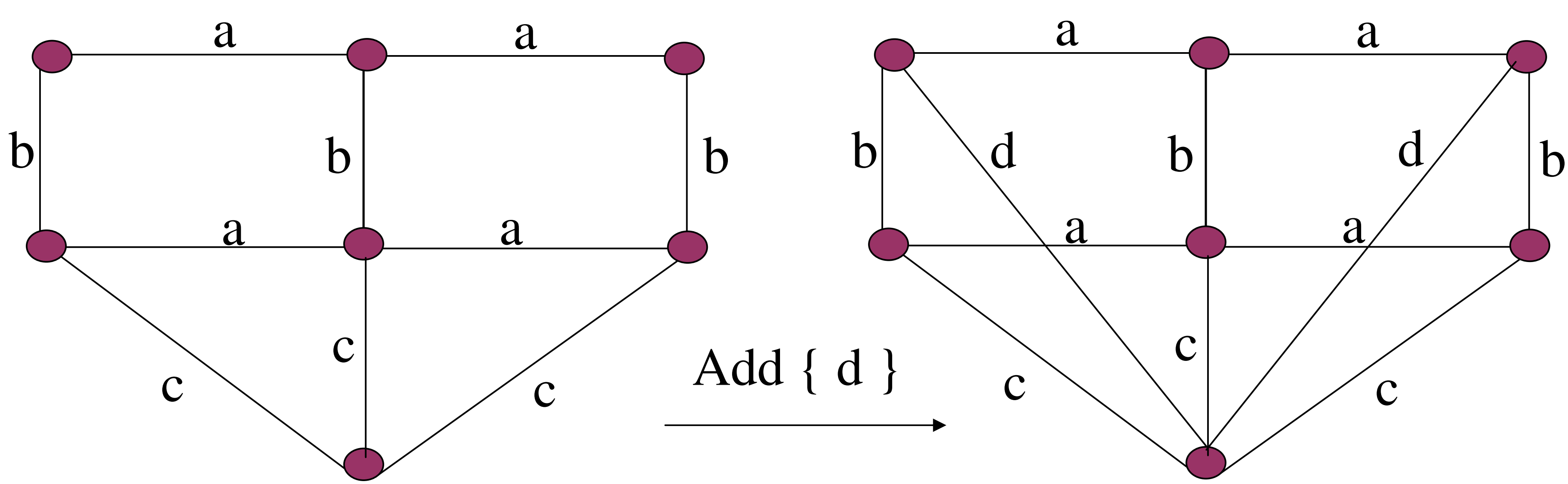
  d. Continue until no longer possible

  e. Call this solution T and output T

An Example of Mutation

S = { a, b, c }

S = { a, b, c, d }



Ordering: a, b, c, d

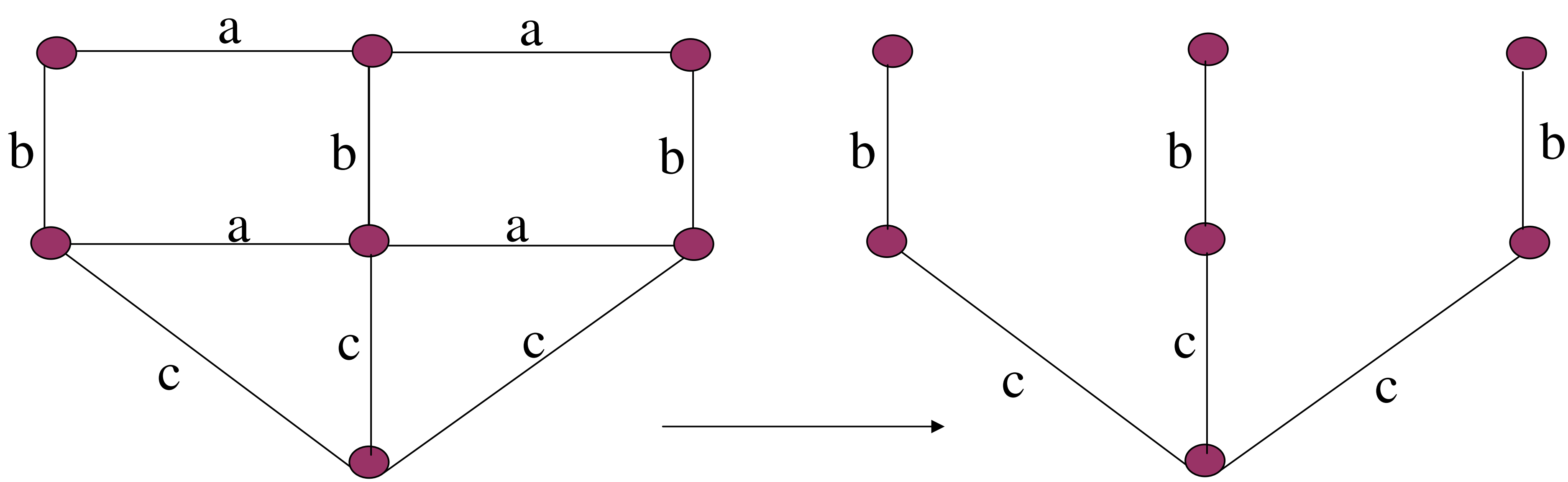
An Example of Mutation

Remove { d }

S = { a, b, c }

Remove { a }

S = { b, c }



T = { b, c }

Computational Results

- 81 combinations: n = 20 to 200 / l = 20 to 250 / density = .2, .5, .8 / p = 20
- 20 sample graphs for each combination
- The average number of labels is compared
- GA beats MVCA in 53 of 81 cases (16 ties, 12 worse)
- No instance required more than 2 seconds CPU time on a Pentium 4 PC with 1.80 GHz and 256 MB RAM

Conclusions & Future Work

- The GA is fast, conceptually simple, and powerful
- It contains a single parameter
- We think GAs can be applied successfully to a host of other NP – hard problems
- Future work
  - More extensive computational testing
  - Add edge costs
  - Examine other variants