1. (50 points) An electromagnetic wave of frequency $\omega$ is normally incident on an infinite conducting surface located at $z = 0$ (see the figure above). Assume that $\mathbf{J} = \sigma \mathbf{E}$ inside of the conductor, where $\sigma$ is a known constant.

(a) Starting from Maxwell's equations calculate the dispersion relation for the electromagnetic wave in the vacuum region and in the conductor. Assume large but finite conductivity $\sigma$ to simplify the latter.

(b) What are the boundary conditions on the fields at the conducting surface?

(c) Calculate the complex amplitude of the transmitted and reflected waves ($k_0 \sigma \ll 1$).

(d) Calculate the Poynting vector $\mathbf{S}$ of the incident wave, reflected wave and transmitted wave at the boundary ($z = 0$). What fraction of the energy flux is transmitted into the conductor? What happens to this energy?

2. (50 points) An infinite cylindrical rod (radius $a$) of magnetically permeable material $\mu$ is placed in an external magnetic field $\mathbf{B} = B_0 \mathbf{\hat{x}}$.

(a) Sketch the magnetic field lines for the case $\mu \gg \mu_0$.

(b) What are the conditions that must be satisfied by $\mathbf{B}$ and $\mathbf{H}$ at the boundary?

(c) Calculate $\mathbf{B}$ and $\mathbf{M}$ everywhere for $\rho < a$.

(d) If the cylinder has a finite length $L \gg a$, where would the solutions in (c) be valid? How would the magnetic field produced by the cylinder fall off at distances large compared with $L$? Derive an expression for the magnetic...
field produced by the cylinder that is valid at distances large compared with $L$.

Hint: the solution can be written in terms of the magnetic potential $\phi_m$ which can be obtained as an integral that can be evaluated. You don't have to evaluate $B$ explicitly but make sure that you have an expression from which it can be easily evaluated.
a) calculate dispersion relation for conduction

\[ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \varepsilon_0 \varepsilon \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \varepsilon \mathbf{E} \]

\[ \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \]

\[ \nabla \times \nabla \times \mathbf{E} = -\nabla (\nabla \cdot \mathbf{E}) = -\nabla \cdot \nabla \mathbf{E} + \nabla^2 \mathbf{E} = \varepsilon_0 \varepsilon \mathbf{E} \]

\[ + k^2 = \mu_0 \varepsilon (\omega^2 + i\omega) \]

\[ k = \frac{\mu_0 \omega}{\varepsilon} (1+c) \equiv \frac{1}{\varepsilon} (1+i) \]

\[ \nabla \times \mathbf{B} = \nabla \times \mathbf{E} = \varepsilon_0 \varepsilon \mathbf{E} \]

\[ k^2 = \varepsilon_0 \omega^2 \]
b) Since \( \mathbf{u} = \mathbf{u}_0 \) on both sides,
\[\hat{n} \times \mathbf{B} = 0\]
\[\hat{n} \times \mathbf{E} = 0\]

c) Continuity of \( \mathbf{E}_x \)

1. \( \mathbf{E}_0 - \mathbf{E}_r = \mathbf{E}_c \) 
   continuity of \( \mathbf{B}_y \)

2. \( \mathbf{B}_0 + \mathbf{B}_r = \mathbf{B}_c \) 
   relate \( \mathbf{B}_y \) to \( \mathbf{E}_y \)

   in vacuum:
   \[\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0\]
   \[i \kappa \mathbf{E}_x - \omega \mathbf{B}_y = 0\]
   \[\mathbf{E}_x = \omega \mathbf{B}_y\]

   in conductors:
   \[
   \left( \frac{1}{\kappa} + (1+i) \right) \mathbf{E}_x = \frac{\omega}{\kappa} \mathbf{B}_y
   \]

From 1) and 2)

\[\kappa \mathbf{B}_0 - \kappa \mathbf{B}_r = \omega \mathbf{B}_c \frac{\xi}{(1+i)\kappa} = \kappa \xi \frac{\mathbf{B}_c}{1+i}\]
\[\mathbf{B}_0 + \mathbf{B}_r = \mathbf{B}_c\]
\[2 \mathbf{B}_0 = \mathbf{B}_c \left( 1 + \frac{\xi}{1+i} \right)\]
\[ B_t = \frac{2B_0}{1 + \frac{ks}{1 + t}} \]

\[ B E_t = \frac{\omega 2B_0 S c}{(1 + i) C} \]

\[ E_t = \frac{2ks S cB_0}{1 + t} \]

\[ B_a = B_t - B_0 \Rightarrow B_0 \]

\[ E_r = cB_0 \]

\[ a) \text{ Poynting flux} \]

\[ S_{\text{t}}^t = \text{Re} \frac{E_t B_t^*}{2 \mu_0} = \text{Re} \frac{\omega S}{1 + i} \frac{B_0}{2 \mu_0} \frac{1}{1 + \frac{2}{1 - i}} \]

\[ = \frac{\omega S}{\mu_0 c} B_0^2 c = ks \frac{B_0^2 c}{\mu_0} \]

\[ \text{Vacuum:} \]

\[ S_{\text{t}}^0 = \frac{1}{2 \mu_0} c B_0^2 \]

\[ \frac{S_{\text{t}}^t}{S_{\text{t}}^0} = 2ks \]

\[ \Rightarrow \text{dissipated by Joule heating} \]
(2) (a) \[ \mu \rightarrow \mu_0 \]

\[ \Rightarrow \quad B_\infty \]

\[ \downarrow \quad B_0 \]

\[ \rightarrow \quad B_\infty \]

\[ \Rightarrow \quad \nabla \times \hat{e} \]

\[ \frac{q+e}{a-e} \]

\[ = 0 \quad \Rightarrow \quad H_q \]

\[ \frac{q+e}{a-e} \]

\[ = 0 \quad \Rightarrow \quad H_\infty \]

b) \[ B_\infty \frac{q+e}{a-e} \]

\[ = 0 \quad \Rightarrow \quad B_\infty \]

\[ \frac{q+e}{a-e} \]

\[ = 0 \quad \Rightarrow \quad H_q \]

\[ \frac{q+e}{a-e} \]

\[ = 0 \quad \Rightarrow \quad H_\infty \]

c) \[ \text{Since } \nabla \cdot \mathbf{J} = 0 \quad \Rightarrow \quad \nabla \times \mathbf{H} = 0 \]

\[ \Rightarrow \quad \mathbf{H} = -\nabla \phi \mathbf{m} \]

\[ B_\infty = \mu_0 H_\infty = -\mu_0 \nabla \phi \mathbf{m} \]

\[ \nabla \cdot B = 0 \quad \Rightarrow \quad \nabla \cdot \mu_0 \nabla \phi \mathbf{m} = 0 \]

\[ \text{For } a \neq \infty \quad \Rightarrow \quad \nabla^2 \mathbf{m} = 0 \]

\[ \text{For large } \infty \quad \mathbf{m} = -\frac{B_0 \mathbf{X}}{\mu_0} = -\frac{B_0 \mathbf{e} \cos \phi}{\mu_0} \]

\[ \underline{\text{BCs}} \]

\[ H_q \]

\[ \frac{q+e}{a-e} \]

\[ = 0 \quad \Rightarrow \quad \frac{\partial \phi}{\partial a} \mathbf{m} = 0 \]

\[ B_\infty \]

\[ \frac{q+e}{a-e} \]

\[ = 0 \quad \Rightarrow \quad \mu \frac{\partial \phi}{\partial \phi} \mathbf{m} = 0 \]
\[ e^m \sim e \quad e^m \]

\[ e^m = e^m \cos(m \phi) \left( \frac{a}{e} \right)^m - B \cdot e \cos \phi \frac{\mu}{m_o} \]

\[ \Rightarrow \text{other } e^m \text{ solutions discarded} \]

\[ \Rightarrow \text{symmetric around } \phi = 0 \]

\[ \phi < a \]

\[ e^m = e^m \cos m \phi \left( \frac{e}{a} \right)^m \]

**only** \( m = 1 \) survives matching

\[ e^m > = c_1^> \cos \phi \left( \frac{a}{e} \right) - B_o \cdot e \cos \phi \frac{\mu}{m_o} \]

\[ e^m < = c_1^< \cos \phi \left( \frac{e}{a} \right) \]

\[ \frac{\partial e^m}{\partial e} \text{ matching} \]

1

\[ c_1^> - \frac{B_o a}{m_o} = c_1^< \]

\[ \mu \left( \frac{\partial e^m}{\partial e} \right) \text{ matching} \]

2

\[ C_1^> \mu_0 \frac{1}{e^m} + B_o a = -\mu \left( \frac{1}{m_o} \right) c_1^< \]

**Putting 1 into 2**

\[ c_1^< + \frac{B_o a}{m_o} + \frac{B_o a}{m_o} = -\mu \left( \frac{1}{m_o} \right) c_1^< \]
\[ c_i^2 = -2 \frac{B_0 a}{\mu_0} \frac{1}{1 + \frac{\mu}{\mu_0}} \]

\[ c_i^+ = \frac{B_0 a}{\mu_0} + c_i^- = \frac{B_0 a}{\mu_0} \left( 1 - \frac{2}{1 + \frac{\mu}{\mu_0}} \right) \]

\[ c_i^+ = \frac{B_0 a}{\mu_0} \frac{\mu}{\mu_0} - 1 \frac{\mu}{\mu_0} + 1 \]

\[ P < a \]

\[ c_i^+ = \frac{c_i}{a} - \frac{x}{\frac{\mu}{\mu_0}} \]

\[ H^+ = -c_i \frac{1}{a} \frac{x}{\frac{\mu}{\mu_0}} \]

\[ B^+ = -c_i \frac{1}{a} \frac{\mu}{\mu_0} \frac{x}{\frac{\mu}{\mu_0}} \]

\[ m = \mu_0 \left( \frac{1}{\mu} + \frac{1}{\mu_0} \right) = \frac{\mu}{\mu_0} \]

\[ B_{\mu_0} = \frac{B}{\mu} + \frac{B}{\mu_0} \]

\[ H = \frac{B}{\mu} \left( \frac{1}{\mu_0} - \frac{1}{\mu} \right) = B \frac{\mu - \mu_0}{\mu \mu_0} \]

\[ m^+ = -c_i \left( \frac{\mu}{\mu_0} - 1 \right) \frac{x}{\frac{\mu}{\mu_0}} \]

\[ B^+ = 2 B_0 \frac{\mu}{\mu_0} \frac{1}{1 + \frac{\mu}{\mu_0}} \]

\[ M^+ = 2 B_0 \frac{1}{\mu_0} \left( \frac{\mu}{\mu_0} - 1 \right) \frac{x}{\mu_0 + 1} \]

\[ = \mu_0 \hat{x} \]
d) Finite length with \( L \gg a \).

The solutions are valid for \( \varepsilon \ll L \) and not too close to the ends.

For distances large compared to \( L \) it will look like a magnetic dipole

\[
\nabla \cdot B = 0 = \nabla \cdot \mu_0 \left( - \nabla \epsilon_m + \frac{m}{\mu} \right)
\]

\[
\nabla^2 \epsilon_m = \nabla \cdot \frac{m}{\mu} = -4\pi \left( - \frac{\mu_0 M}{\mu} \right)
\]

\[
\epsilon_m = -\frac{1}{4\pi} \int \frac{\nabla' \cdot \frac{\mu_0 m(x')}{|x-x'|}}{|x-x'|} \, dx'
\]

\[
= -\frac{1}{4\pi} \int \nabla \cdot \frac{m}{|x|} \, dx
\]

\[
\epsilon_m = \frac{\mu_0 m \cdot \mathbf{x}}{|x|^3}
\]

\[
\epsilon_m = \frac{\mu_0 a^2 L}{2}
\]

\[
\nabla \epsilon_m = -\frac{1}{4\pi} \frac{m \cdot \mathbf{x}}{|x|^3}
\]

\[
\nabla \epsilon_m = \left( \frac{\mu_0 a^2 L}{|x|^3} \right)
\]

\[
\frac{m}{\mu} = \frac{\mu_0 a^2 L}{|x|^3}
\]

\[
\nabla \cdot B = -\mu_0 \nabla \epsilon_m
\]