A path $p$ of length $h$ in a directed graph $G = (V,E)$ is an $(h+1)$-tuple of distinct vertices $(x_0, \ldots, x_h) \in V^{h+1}$ such that, for each $i \in \{0, \ldots, h\}$, we have $(x_i, x_{i+1}) \in E$. We say that $x_0$ is the source vertex of the path $p$, denoted $s(p)$, and that $x_h$ is the target vertex of the path $p$, denoted $t(p)$. For $0 \leq n \leq f$, we will write $\mathcal{P}(n) = \mathcal{P}_n$, for the $n^{th}$ vertex in the path $p$. We will write $\mathcal{P}_n$ for the length of the path $p$.

We define paths in this way so we can encode the notion of a trivial path $(x_0)$ consisting of only one vertex for which its length $h = 0$.

Informally, the Generalized Action Graph is defined as follows. First, we let $T_{n,k}$ be the labeled directed tree with one vertex with label 0. Then, inductively, the graph $T_{n,1,k}$ is constructed as follows. For each path $p$ with target vertex with label $n$ in $T_{n,k}$, we adjoin $(\ell+1)$ new vertices to the source vertex of the path $p$ in $T_{n,k}$. We label these new vertices $n+1$.

Below is an example of the first few action graphs with $k = 2$.

![Example of an Labeled Directed Graph](image)

### Generalized Action Graphs

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### Some Essential Properties

- All the generalized action graphs $T_{n,k}$ are trees (that is, they contain no cycles). As a result, it makes sense to think of $T_{n,k}$ as a rooted tree with root vertex the sole vertex labeled 0.
- Resolving $T_{n,k}$ as a directed tree, in the graph $T_{n,k}$, any rooted subtrees with root vertex labeled 0 and containing all the children of that vertex are isomorphic to $T_{n,k}$. So in a sense the generalized action graphs are self-similar. The visual below illustrates this property.

![An example path subdivision by an order-preserving map](image)

The main result we proved was the following:

In the generalized action graphs $T_{n,k}$, the number of vertices labeled $n$ is the Fuss-Catalan number $C_{n,k}$.

We proved this in two ways:
- Using a counting argument, we proved this result more or less directly.
- We established a natural bijection between the vertices in $T_{n,k}$ and the collection of full $(k+1)$-ary trees with at most $n(k+1) + 1$ vertices. Both methods rely on the following elegant recharacterization of the addition of $C_{n+1,k}$ vertices for any given path $p$. This is the number of order-preserving maps $g : \{1, \ldots, f(p)\} \to \{1, \ldots, k\}$.

Here is the idea for the proof of the first method. For any vertex of label $n$, we can associate with it the $k$-ary tree created from the action graph $T_{n,k-1}$ and the $n+1$ new vertices produced by the order-preserving map $g : \{1, \ldots, f(p)\} \to \{1, \ldots, k\}$.

![An example path subdivision by an order-preserving map](image)

From the path subdivision, we can deduce from the self-similarity of the generalized action graphs that there are $\prod_{i=1}^{f(p)} (n+1)$ paths with the same labeling and order-preserving map. Summing over all these paths, we get the recurrence:

$$f(n,k) = \sum_{n_0 + \cdots + n_{f(p)-1} = n-1} \prod_{i=1}^{f(p)} (n_i + 1),$$

which is exactly the recurrence for the Fuss-Catalan numbers $C_{n,k}$.

The main idea of the bijection is as follows. First, note that for each vertex $v$ with label $n$ in action graph $T_{n,k}$, there is a unique path from 0 to $v$. If we consider the set of complete $(k+1)$-ary trees with $n(k+1) + 1$ nodes labelled by preorder traversal, each tree will be of the form

$$\mathcal{F}(n,k) = \prod_{i=1}^{f(p)} (n_i + 1),$$

where $m = n(k+1) + 1$ and every node on the path from 0 to $v$ is of the form $a(k+1)$, with $a \in \mathbb{N}$ and $a < n$. Additionally, each triangle represents $k$ complete $(k+1)$-ary trees.

If we take this path from 0 to $m$ in each of our trees and divide the label of each vertex by $(k+1)$, we are able to find a unique path in our action graph $T_{n,k}$ to which it corresponds.