HW 5

(1) (i) if $a \in A^c$
$\exists$ a sequence $\{a_k\}$ such that $a_k \in A \quad \forall k \in \mathbb{N}$, and $\lim_{k\to\infty} a_k = a$
since $A \subset B$, we have $a_k \in B \quad \forall k \in \mathbb{N}$
$\therefore a \in B^c$
$\Rightarrow A^c \subset B^c$

(ii) first we prove $(A \cup B)^c \subset A^c \cup B^c$
let $x \in (A \cup B)^c$,
$\exists$ a sequence $\{x_k\}$ such that $x_k \in A \cup B \quad \forall k \in \mathbb{N}$, and $\lim_{k\to\infty} x_k = x$
$x_k \in A \cup B$, so $x_k \in A$ or $x_k \in B$
if $x_k \in A$, then relabel it x'_k, if not, then $x_k \in B$, relabel it x''_k
we have $\{x'_k\} \cup \{x''_k\} = \{x_k\}$ and $\{x'_k\} \cap \{x''_k\} = \varnothing$,
Since $\{x_k\}$ is infinite, one of $\{x'_k\}$ and $\{x''_k\}$ has to be an infinite subsequence of $\{x_k\}$.
Suppose $\{x'_k\}$ is the infinite subsequence, then by
$\{x'_k\} \subset A$ and $\lim_{k\to\infty} x'_k = \lim_{k\to\infty} x_k = x$,
we get $x \in A^c$.
Otherwise $\{x''_k\}$ would be the infinite subsequence,
and by the same arguments, we get $x \in B^c$
Thus, we have proved that if $x \in (A \cup B)^c$ then $x \in A^c$ otherwise B^c
$\Rightarrow (A \cup B)^c \subset A^c \cup B^c$.
Now we prove the other direction, $A^c \cup B^c \subset (A \cup B)^c$
by (i),
$A \subset A \cup B \Rightarrow A^c \subset (A \cup B)^c$
$B \subset A \cup B \Rightarrow B^c \subset (A \cup B)^c$
$\Rightarrow (A^c \cup B^c) \subset (A \cup B)^c$

$\Rightarrow (A^c \cup B^c) = (A \cup B)^c$

(iii) by (i)
$(A \cap B) \subset A \Rightarrow (A \cap B)^c \subset A^c$
$(A \cap B) \subset B \Rightarrow (A \cap B)^c \subset B^c$
$\Rightarrow (A \cap B)^c \subset A^c \cap B^c$

(2) Since $a_k \in I_k \Rightarrow a - \frac{1}{2^k} < a_k < a + \frac{1}{2^k} \Rightarrow |a - a_k| < \frac{1}{2^k}$
$\forall \epsilon$, $\exists N$ such that $2^k > \frac{1}{\epsilon} \quad \forall k > N$,
then $|a - a_k| < \frac{1}{2^k} < \epsilon \quad \forall k > N$
$\lim_{k\to\infty} a_k = a$

(3) $a_i = b_{(i,j_i)}$
then $|a_i - a| = |b_{(i,j_i)} - a| \leqslant |b_{(i,j_i)} - b_i| + |b_i - a| < 2|b_i - a|$
Since $b_i \to a$
given $\epsilon > 0$
$\exists N_\epsilon$ such that $|b_i - a| < \frac{\epsilon}{2}$ for $i > N_\epsilon$
then for the same N_ϵ, $|a_i - a| < \epsilon$ for all $i > N_\epsilon$
$\Rightarrow \lim_{i\to\infty} a_i = a$

(4) A, B are closed, so $A = A^c$, $B = B^c$

(i) By problem 1 and the fact that A, B are closed,
$(A \cap B)^c \subset A^c \cap B^c \subset A \cap B$
Since $(A \cap B)^c \supset A \cap B$
$(A \cap B)^c = (A \cap B)$
$A \cap B$ is closed.

(ii) Again from problem 1,
$(A \cup B)^c = A^c \cup B^c = A \cup B \Rightarrow A \cup B$ is closed.

(iii) from (ii), we get inductively that finite intersection of closed sets is closed;
$(\bigcap_{k=1}^{N} A_k)^c = (\bigcap_{k=1}^{N} A_k)$ for all A_k closed

for any $N \in \mathbb{N}$
$\bigcap_{k=1}^{\infty} A_k \subset \bigcap_{k=1}^{N} A_k$
$\therefore (\bigcap_{k=1}^{\infty} A_k)^c \subset (\bigcap_{k=1}^{N} A_k)^c = \bigcap_{k=1}^{N} A_k \subset A_i \quad \forall i \leqslant \mathbb{N}$
Since N can be arbitrarily chosen
$(\bigcap_{k=1}^{\infty} A_k)^c \subset A_i$ for all $i \in \mathbb{N}$
$\Rightarrow (\bigcap_{k=1}^{\infty} A_k)^c \subset \bigcap_{k=1}^{\infty} A_k$

(5) A is dense in $D \Rightarrow D \subset A^c$
since $A \subset B \Rightarrow A^c \subset B^c$
$\Rightarrow C \subset D \subset A^c \subset B^c$
$\Rightarrow B$ is dense in C.

(6) A is unbounded, suppose it is unbounded above
$\Rightarrow$ for every $N \in \mathbb{N}$, there exists $x \in A$ such that $x > N$
We can construct a sequence such that $x_1 > 1$, $x_k > x_{k-1} \quad \forall k > 1$
$\{x_k\}$ is an increasing divergent sequence.
If A is unbounded below, we can find a decreasing divergent sequence analogously.
Since $\{x_k\}$ is a monotone divergent sequence, all of its subsequences are monotone and divergent.
A is not sequentially compact.

(7)

$$f(x) = \begin{cases} \frac{1}{q} & \text{for} \quad x = \frac{p}{q} \quad \text{rational} \quad \text{(p,q in simplest form)} \\ 0 & \text{for} \quad x \quad \text{irrational} \end{cases}$$

Given $x \in \mathbb{R}\backslash\mathbb{Q}$, and $\epsilon > 0$
$\exists k \in \mathbb{N}$, such that $k > \frac{1}{\epsilon}$
for rational $\frac{p}{q} \in (x - \frac{1}{2k}, x + \frac{1}{2k})$, and $|\frac{1}{q}| > \frac{1}{k}$,
we have $\cdots < \frac{p-2}{q} < \frac{p-1}{q} < x - \frac{1}{2k} < \frac{p}{q} < x + \frac{1}{2k} < \frac{p+1}{q} < \frac{p+2}{q} \cdots$
in other words, if $|q| < k$, there is only one possible p such that $\frac{p}{q}$ lies in the interval $(x - \frac{1}{2k}, x + \frac{1}{2k})$
therefore, there are only a finite number of rational number $\frac{p}{q}$ in $(x - \frac{1}{2k}, x + \frac{1}{2k})$ such that $|\frac{1}{q}| > \frac{1}{k}$,
let δ be the minimum of the distances of these $\frac{p}{q}$ from x.
then there does not exist any rational number $\frac{p}{q}$ such that $|\frac{1}{q}| > \frac{1}{k}$ in $(x - \delta, x + \delta)$
$\Rightarrow$ for all rational $\frac{p}{q}$ in $(x - \delta, x + \delta)$, we have $f(\frac{p}{q}) = \frac{1}{q} < \frac{1}{k} < \epsilon$
Since for irrational $y \in (x - \delta, x + \delta)$, $f(y) = 0$
for all $y \in (x - \delta, x + \delta)$, we have $f(y) < \epsilon$

δ can be chosen for any given ϵ
$\Rightarrow f$ is continuous at x irrational.