1. 4.3 Problem 1

   a. False.
   Let \( f(x) = x^3 \), \( f(x) \) is strictly increasing, but \( f'(0) = 0 \)

   b. True.
   Since \( f \) is nondecreasing, we have \( f(x) \leq f(y) \) if \( x < y \).
   So \( \frac{f(x) - f(x_0)}{x - x_0} \geq 0 \) for all \( x, x_0 \in \mathbb{R} \)
   By Lemma 2.21, \( \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \geq 0 \)
   \( f'(x_0) \geq 0 \) for all points \( x_0 \) in \( \mathbb{R} \)

   c. True.
   Since \( f(0) \geq f(x) \) for all \( x \in [-1, 1] \),
   \( \frac{f(x) - f(0)}{x - 0} \leq 0 \) for \( x > 0 \)
   So \( \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} \leq 0 \)
   \( \frac{f(x) - f(0)}{x - 0} \geq 0 \) for \( x < 0 \)
   So \( \lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} \geq 0 \)
   Since derivative exists, both limits are equal.
   \( f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = 0 \)

   d. False
   Let \( f(x) = x \)
   \( f(1) \geq f(x) \) for all \( x \in [-1, 1] \)
   but \( f'(1) = 1 \neq 0 \)

2. 4.3 Problem 4

   \( f'(x) = 3x^2 - 3 = 3(x^2 - 1) < 0 \) for \( 0 < x < 1 \)
   \( \Rightarrow f(x) > f(y) \) for all \( 0 < x < y < 1 \)
   If \( f(x) \) has two solutions in \( (0, 1) \), then we would have \( f(x) = f(y) = 0 \) for some \( 0 < x < y < 1 \), which contradicts the previous statement.

3. 4.3 Problem 7

   You can use Rolle’s Theorem as follows.
   \( f'(x) = nx^{n-1} + a \)
   \( f''(x) = n(n-1)x^{n-2} > 0 \) for all \( x \in \mathbb{R} \) since \( n \) is even \( \Rightarrow f' \) is strictly increasing.
   If \( f \) has three or more zeros,
   \( \exists a, b, c \) such that \( f(a) = f(b) = f(c) = 0 \).
   Then by Rolle’s Theorem \( \exists x \in (a, b) \) and \( y \in (b, c) \)
   such that \( f'(x) = 0 \) and \( f'(y) = 0 \)
   But \( f' \) cannot have two distinct zeros since \( f' \) is strictly increasing.
   Therefore, \( f \) has at most two zeros.

   If \( n \) is odd, there could be one or three solutions depending on the values of \( a, b \).
   If \( a > 0 \), then \( f'(x) > 0 \) for all \( x \), \( f(x) \) is strictly increasing. So \( f(x) \) has exactly one solution in this case.
   If \( a = 0 \), then \( f(x) = x^n + b \) is strictly increasing, \( f(x) \) has only one solution.
   If \( a < 0 \) then \( f(x) \) is increasing in the intervals \( (-\infty, -(\frac{a}{n})^{\frac{1}{n-1}}) \) and \( ((\frac{a}{n})^{\frac{1}{n-1}}, \infty) \),
   decreasing in the interval \( -(\frac{a}{n})^{\frac{1}{n-1}}, (\frac{a}{n})^{\frac{1}{n-1}}) \).
   So depending on \( b \), \( f(x) \) can have one or three solutions.
4. 4.3 Problem 11
Suppose the contrary. (Suppose \( f \) has \( n + 1 \) or more solutions)
Then \( \exists a_1 < a_2 < \cdots < a_{n+1} \) such that
\[
\begin{align*}
f(a_1) &= f'(a_2) = \cdots = f(a_{n+1}) = 0
\end{align*}
\]
Then by Rolle’s theorem, there exists \( x_1, x_2, \ldots, x_n \)
such that \( a_1 < x_1 < a_2 < x_2 < a_3 < \cdots < a_n < x_n < x_{n+1} \) and
\[
\begin{align*}
f'(x_1) &= f'(x_2) = \cdots = f'(x_n) = 0,
\end{align*}
\]
but this contradicts to the fact that \( f' \) has at most \( n - 1 \) zeros.
So \( f \) can have at most \( n \) solutions (zeros).

5. 4.4 Problem 3

a. \( f''(t) = 2t, \ g'(t) = 3t^2 \)
\[
\begin{align*}
f(1)-f(0) &= 0, \quad g(1)-g(0) = ?
f'(c) &= 0, \quad g'(c) = ?
\end{align*}
\]
so if \( c = \frac{2}{3}, \) we have
\[
\begin{align*}
f(1)-f(0) &= f'(c)(1-0), \quad \Rightarrow c = \frac{1}{\sqrt{3}}
\end{align*}
\]
There is no \( c \) that satisfies both equations.

b. If \( f(1) - f(0) = f'(c)(1-0), \) then \( f'(c) = 1, \Rightarrow c = \frac{1}{\sqrt{3}} \)
If \( g(1) - g(0) = g'(0)(1-0), \) then \( g'(c) = 1, \Rightarrow c = \frac{1}{\sqrt{3}} \)

6. 4.4 Problem 5
By Theorem 4.24, and the condition \( f(0) = f'(0) = \cdots = f^{(n-1)}(0) = 0 \)
for any \( x \neq 0, \) there is a point \( z \) strictly between \( x \) and \( 0 \) such that
\[
\begin{align*}
f(x) &= \frac{f^{(n)}(z)}{n!}(x)^n
\end{align*}
\]
Since \( f^{(n)} \) is bounded, \( \exists N \) such that \( |f^{(n)}(x)| < N \) for all \( x \in (-1, 1) \)
\[
\begin{align*}
|f(x)| &= |\frac{f^{(n)}(z)}{n!}(x)^n| \leq M|x|^n \quad \text{where} \ M = \frac{N}{n!}.
\end{align*}
\]

7. 4.4 Problem 7

solution 1
Let \( g(h) = f(x_0 + h) - 2f(x_0) + f(x_0 - h), \) then
\[
\begin{align*}
g'(h) &= f'(x_0 + h) - f'(x_0 - h) \
g''(h) &= f''(x_0 + h) + f''(x_0 - h) \
g(0) &= 0, \ g'(0) = 0
\end{align*}
\]
By theorem 4.24, or Lagrange Remainder Theorem,
for each \( h \) there is a \( z = z(h) \in (0, h) \) such that
\[
\begin{align*}
g(h) &= \frac{g''(z)}{2!}h^2 \
\Rightarrow f(x_0 + h) - 2f(x_0) + f(x_0 - h) &= g(h) = \frac{g''(z)}{2!}h^2 \
\Rightarrow \lim_{h \to 0} f(x_0 + h) - 2f(x_0) + f(x_0 - h) &= \lim_{h \to 0} \frac{g''(z)}{2} = \lim_{h \to 0} \frac{f''(x_0 + z) + f''(x_0 - z)}{2} = f''(x_0)
\end{align*}
\]
The last equality is true because \( z(h) \to 0 \) as \( h \to 0. \)

solution 2
Since \( \lim_{h \to 0} f(x_0 + h) - 2f(x_0) + f(x_0 - h) = 0 \) and
\( \lim_{h \to 0} h^2 = 0, \)
by L’hopital’s rule,
\[
\begin{align*}
\lim_{h \to 0} f(x_0 + h) - 2f(x_0) + f(x_0 - h) &= \lim_{h \to 0} f'(x_0 + h) - f'(x_0 - h) \
\text{again we have,} \ &\lim_{h \to 0} f'(x_0 + h) - f'(x_0 - h) = 0 \text{ and}
\end{align*}
\]
lim_{h \to 0} 2h = 0
By L’hopital’s rule
\[ \Rightarrow \lim_{h \to 0} \frac{f'(x_0+h) - f'(x_0-h)}{2h} = \lim_{h \to 0} \frac{f''(x_0+h) + f''(x_0-h)}{2} = f''(x_0) \]
\[ \Rightarrow \lim_{h \to 0} \frac{f(x_0+h) - 2f(x_0) + f(x_0-h)}{h^2} = \frac{f''(x_0) + f''(x_0)}{2} = f''(x_0) \]

8. 8.1 Problem 2

a. \( f(x) = \int_0^x \frac{1}{1+t^2} dt \)
\[ f'(x) = \frac{1}{1+x^2} \]
\[ f''(x) = \frac{-2x}{(1+x^2)^2} \]
\[ f'''(x) = \frac{-2}{(1+x^2)^2} + \frac{8x}{(1+x^2)^3} \]
\[ p_3(x) = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 \]
\[ = x - \frac{2}{3!} x^3 \]
\[ = x - \frac{1}{3} x^3 \]

b. \( f(x) = \sin x \)
\[ f'(x) = \cos x \]
\[ f''(x) = -\sin x \]
\[ f'''(x) = -\cos x \]
\[ p_3(x) = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 \]
\[ = x - \frac{1}{3!} x^3 \]
\[ = x - \frac{1}{6} x^3 \]

c. \( f(x) = \sin x + x^{200} \)
\[ f'(x) = \cos x + 200x^{199} \]
\[ f''(x) = -\sin x + (200)(199)x^{198} \]
\[ f'''(x) = -\cos x + (200)(199)(198)x^{197} \]
\[ p_3(x) = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 \]
\[ = x - \frac{1}{3!} x^3 \]
\[ = x - \frac{1}{6} x^3 \]

d. \( f(x) = \sqrt{2-x} \)
\[ f'(x) = \frac{-1}{2(2-x)^{\frac{3}{2}}} \]
\[ f''(x) = -\frac{1}{4(2-x)^{\frac{5}{2}}} \]
\[ f'''(x) = -\frac{3}{8(2-x)^2} \]

\[
p_3(x) = f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3
\]
\[
= 1 - \frac{1}{2}(x-1) - \frac{1}{2!} \cdot \frac{1}{4}(x-1)^2 - \frac{1}{3!} \cdot \frac{3}{8}(x-1)^3
\]
\[
= 1 - \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 - \frac{1}{16}(x-1)^3
\]

9. 8.1 Problem 4
Since \( p_3(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 \)
we know that \( f(0) = 1, f'(0) = 4, f''(0) = -2 \)
Since \( f \) has three derivatives, \( \Rightarrow f, f', \) and \( f'' \) are continuous.
\( \exists \delta_1 > 0, \delta_2 > 0, \) and \( \delta_3 > 0 \) such that
\[
|f(x) - f(0)| < \frac{1}{2} \text{ for } |x| < \delta_1
\]
\[
|f'(x) - f'(0)| < \frac{1}{2} \text{ for } |x| < \delta_2
\]
\[
|f''(x) - f''(0)| < \frac{1}{2} \text{ for } |x| < \delta_3
\]
\( \Rightarrow \text{ for } |x| < \delta = \min\{\delta_1, \delta_2, \delta_3\} \)
\[
f(x) > f(0) - \frac{1}{2} = \frac{1}{2} > 0
\]
\[
f'(x) > f'(0) - \frac{1}{2} = 4 - \frac{1}{2} > 0
\]
\[
f''(x) < f''(0) + \frac{1}{2} = -2 + \frac{1}{2} < 0
\]
Hence \( f \) is positive for \( |x| < \delta, \)
\( f' > 0 \text{ for } |x| < \delta, \) which implies \( f \) is strictly increasing for \( |x| < \delta \)
\( f'' < 0 \text{ for } |x| < \delta, \) which implies \( f' \) is strictly decreasing for \( |x| < \delta. \)

10. 8.2 Problem 2
\[
f(x) = (1 + x)^{\frac{1}{3}}
\]
\[
f'(x) = \frac{1}{3}(1 + x)^{-\frac{2}{3}}
\]
\[
f''(x) = \frac{2}{9}(1 + x)^{-\frac{5}{3}}
\]
\[
f'''(x) = \frac{10}{27}(1 + x)^{-\frac{8}{3}}
\]
By the Lagrange remainder theorem, for each \( x > 0 \) there exists \( c_x \in (0, x) \) such that
\[
f(x) = f(0) + f'(0)x + \frac{f''(c_x)}{2!}x^2
\]
\[
f''(c_x)x^2 = -\frac{2}{9}(1 + c_x)^{-\frac{5}{3}}x^2 < 0 \text{ for each } x > 0
\]
\( \Rightarrow \) \( f(x) < f(0) + f'(0)x = 1 + \frac{1}{3} \)

Again by the Lagrange remainder theorem, for each \( x > 0 \) there exists \( d_x \in (0, x) \) such that
\[
f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(d_x)}{3!}x^3
\]
\[
f'''(d_x)x^3 = \frac{10}{27}(1 + d_x)^{-\frac{8}{3}}x^3 > 0 \text{ for each } x > 0
\]
\( \Rightarrow \) \( f(x) > f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 = 1 + \frac{5}{3} \frac{x^2}{3} \)
\[
\Rightarrow 1 + \frac{5}{3} - \frac{x^2}{9} < (1 + x)^{\frac{1}{3}} < 1 + \frac{5}{3} \text{ for } x > 0
\]

11. 8.2 Problem 8
\((\Rightarrow)\)
Suppose that \( x_0 \) is a root of order \( k \) of the polynomial \( p \)
then \( p(x) = (x - x_0)^k r(x), \) where \( r(x_0) \neq 0 \)
Differentiating directly,
\[ p'(x) = k(x - x_0)^{k-1}r(x) + (x - x_0)^k r'(x) \]
\[ p''(x) = k(k-1)(x - x_0)^{k-2}r(x) + 2k(x - x_0)^{k-1}r'(x) + (x - x_0)^k r''(x) \]
\[ \vdots \]
\[ p^{(k-1)}(x) = \sum_{i=0}^{k-1} \binom{k-1}{i} r^{(i)}(x) \]
\[ p^{(k)}(x) = \sum_{i=0}^{k} \binom{k}{i} r^{(i)}(x) \]

For \( p(x) \), \( p'(x) \), to \( p^{(k-1)}(x) \), all the terms are multiples of \((x - x_0)\), so \( p(x_0) = p'(x_0) = \cdots = p^{(k-1)}(x_0) = 0 \)
for \( p^{(k)}(x) \), all terms except the term \( k!r(x) \) are multiples of \((x - x_0)\).
So \( p^{(k)}(x_0) = k!r(x_0) \neq 0 \).

(\( \Leftarrow \))

Suppose \( p(x) \) is a polynomial of degree \( n \).
Then the \( n \)th Taylor polynomial for \( p \) at \( x_0 \) is \( p(x) \) itself.

\[ p(x) = \sum_{l=0}^{n} \frac{p^{(l)}(x_0)}{l!} (x - x_0)^l \]

Since \( p(x_0) = p'(x_0) = \cdots = p^{(k-1)}(x_0) = 0 \),
\[ p(x) = \sum_{l=k}^{n} \frac{p^{(l)}(x_0)}{l!} (x - x_0)^l = (x - x_0)^k \sum_{l=0}^{n-k} \frac{p^{(l+k)}(x_0)}{(l+k)!} (x - x_0)^l \]

let \( r(x) = \sum_{l=0}^{n-k} \frac{p^{(l+k)}(x_0)}{(l+k)!} (x - x_0)^l \)
then we know that \( r(x_0) = p^{(k)}(x_0) \neq 0 \)
Therefore we have \( p(x) = (x - x_0)^k r(x) \), where \( r(x_0) \neq 0 \)
So \( x_0 \) is a root of \( p \) with order \( k \).

12. 8.2 Problem 11

a. Since \( f^{(n+1)}(x) \) is continuous and \( f^{(n+1)}(x_0) > 0 \),
there exists \( \delta > 0 \) such that \( f^{(n+1)}(x) > 0 \) for \( x \) in \(|x_0 - x| < \delta\).
By the Lagrange remainder theorem, for each \( x \neq x_0 \) with \(|x - x_0| < \delta\) there is a \( c_x \) strictly between \( x_0 \) and \( x \) satisfying
\[ f(x) = f(x_0) + f'(x_0)(x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n + \frac{f^{(n+1)}(c_x)}{(n+1)!} (x - x_0)^{n+1} \]
Since \( f^{(k)}(x) = 0 \) for \( 1 \leq k \leq n \),
\[ f(x) = f(x_0) + \frac{f^{(n+1)}(c_x)}{(n+1)!} (x - x_0)^{n+1} \]

For \(|x - x_0| < \delta\), we have \(|x_0 - c_x| < |x_0 - x| < \delta\), so we have \( f^{(n+1)}(c_x) > 0 \),
also \( n + 1 \) is even gives \((x - x_0)^{n+1} > 0\)
\[ \Rightarrow \frac{f^{(n+1)}(c_x)}{(n+1)!} (x - x_0)^{n+1} > 0 \]
\[ \Rightarrow f(x) = f(x_0) + \frac{f^{(n+1)}(c_x)}{(n+1)!} (x - x_0)^{n+1} > f(x_0) \Rightarrow x_0 \text{ is a local minimizer.} \]

b. Since \( f^{(n+1)}(x) \) is continuous and \( f^{(n+1)}(x_0) < 0 \),
there exists \( \delta > 0 \) such that \( f^{(n+1)}(x) < 0 \) for \( x \) in \(|x_0 - x| < \delta\).
By the Lagrange remainder theorem, for each \( x \neq 0 \) there is a \( c_x \) strictly between \( x_0 \) and \( x \) satisfying
\[ f(x) = f(x_0) + f'(x_0)(x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n + \frac{f^{(n+1)}(c_x)}{(n+1)!} (x - x_0)^{n+1} \]
Since \( f^{(k)}(x) = 0 \) for \( 1 \leq k \leq n \),
\[ f(x) = f(x_0) + \frac{f^{(n+1)}(c_x)}{(n+1)!} (x - x_0)^{n+1} \]

For \(|x - x_0| < \delta\), we have \(|x_0 - c_x| < |x_0 - x| < \delta\), so we have \( f^{(n+1)}(c_x) < 0 \),
also \( n + 1 \) is even gives \((x - x_0)^{n+1} > 0\)
\[ \Rightarrow \frac{f^{(n+1)}(c_x)}{(n+1)!} (x - x_0)^{n+1} > 0 \]
\[ \Rightarrow f(x) = f(x_0) + f'(x_0)(x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(n+1)}(c_x)}{(n+1)!}(x - x_0)^{n+1} \]

\( f(x) \Rightarrow x_0 \) is a local maximizer.

c. Suppose \( f^{(n+1)}(x_0) > 0 \),
since \( f^{(n+1)}(x) \) is continuous,
there exists \( \delta > 0 \) such that \( f^{(n+1)}(x) > 0 \) for \( x \) in \( |x_0 - x| < \delta \).
By the Lagrange remainder theorem, for each \( x \neq 0 \) there is a \( c_x \) strictly between \( x_0 \) and \( x \) satisfying
\[ f(x) = f(x_0) + f'(x_0)(x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(n+1)}(c_x)}{(n+1)!}(x - x_0)^{n+1} \]
Since \( f^{(k)}(x) = 0 \) for \( 1 \leq k \leq n \),
\[ f(x) = f(x_0) + \frac{f^{(n+1)}(c_x)}{(n+1)!}(x - x_0)^{n+1} \]
For \( |x - x_0| < \delta \), we have \( |x_0 - c_x| < |x_0 - x| < \delta \), so we have \( f^{(n+1)}(c_x) > 0 \)
Since \( n + 1 \) is odd, for \( x > x_0 \),
\[ f(x) = f(x_0) + \frac{f^{(n+1)}(c_x)}{(n+1)!}(x - x_0)^{n+1} > f(x_0) \]
for \( x < x_0 \),
\[ f(x) = f(x_0) + \frac{f^{(n+1)}(c_x)}{(n+1)!}(x - x_0)^{n+1} < f(x_0) \]
\( \Rightarrow x_0 \) is not local minimizer nor a local maximizer.
The case where \( f^{(n+1)}(x_0) < 0 \) is similar.

13. 8.2 Problem 12

a. By the Lagrange remainder theorem, for each \( x \neq 0 \), there exists \( c_h \) strictly between \( x_0 \) and \( x_0 + h \), such that
\[ f(x_0 + h) = f(x_0) + f'(x_0)h + \frac{f''(c_h)}{2!}h^2 \]
Since \( f''(x) > 0 \), \( f'(x) \) is strictly increasing and is one-to-one, so \( c_h \) is unique.
let \( \theta(h) = \frac{x_0 - x}{h} \), clearly, \( 0 < \theta(h) < 1 \), and
\[ f(x_0 + h) = f(x_0) + f'(x_0)h + \frac{f''(x_0 + \theta(h)h)}{2!}h^2 \]

b. By the Lagrange remainder theorem, for each \( h \neq 0 \) there exists \( d_h \) strictly between \( x_0 \) and \( x_0 + h \) such that
\[ f(x_0 + h) = f(x_0) + f'(x_0)h + \frac{f''(x_0)}{2!}h^2 + \frac{f'''(d_h)}{3!}h^3 \]
So with the equation from (a.), we have
\[ \frac{f'''(x_0 + \theta(h)h)}{3!}h^2 = \frac{f''(x_0)h^2}{2!} + \frac{f'''(d_h)}{3!}h^3 \]
Since \( h \neq 0 \)
\[ \Rightarrow \frac{f'''(x_0 + \theta(h)h)}{3!} - \frac{f'''(x_0)}{3!}h \]
\[ \Rightarrow \lim_{h \to 0} \left( \frac{f'''(x_0 + \theta(h)h)}{3!} - \frac{f'''(x_0)}{3!} \right) = \lim_{h \to 0} \frac{f'''(d_h)}{3!} = \frac{f'''(x_0)}{3!} > 0 \]
\[ \Rightarrow \left( \lim_{h \to 0} \theta(h) \right) \left( \lim_{h \to 0} \frac{f''(x_0 + \theta(h)h) - f''(x_0)}{\theta(h)h} \right) = \left( \lim_{h \to 0} \theta(h) \right) f''(x_0) = \frac{f'''(x_0)}{3!} > 0 \]
\[ \Rightarrow \lim_{h \to 0} \theta(h) = \frac{1}{3} \]

14. 8.3 Problem 1

a. \( f(x) = \sin x \)
\[ f'(x) = \cos x \]

;
\[ |f^{(n)}(x)| \leq 1 \text{ for all } n, \text{ and all } x \]
By theorem 8.14,
\[ \sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \]
So we know that for every \( x \), the Taylor series converges.

b. \( f(x) = \cos x \)
\[ f'(x) = -\sin x \]
\[ f''(x) = \cos x \]
\[ f^{(n)}(x) = \cos x \]
\[ f^{(n)}(x) = \cos x \]
\[ f^{(n)}(x) = \cos x \]
\[ f^{(n)}(x) = \cos x \]
By theorem 8.14,
\[ \cos x = \sum_{k=0}^{\infty} \frac{(-1)^{(k+1)}}{(2k)!} (x - \pi)^{2k} \]
So we know that for every \( x \), the Taylor series converges.