1. [10] Suppose that $a \in \mathbb{R}$ has the property that $a < 1/k$ for every $k \in \mathbb{Z}_+$. Show that $a \leq 0$.

Solution: Suppose $a \leq 0$ does not hold. Then by trichotomy $a > 0$. By the Archimedean Property there exists $n \in \mathbb{Z}_+$ such that $1 < na$. Then $1/n < a$, which contradicts the property that $a < 1/k$ for every $k \in \mathbb{Z}_+$. Therefore $a \leq 0$ holds. \hfill \Box

An alternative solution that uses more advanced machinery (and therefore is not as good) is the following. Because constant sequences converge while $1/k \to 0$ as $k \to \infty$, and because of the way limits preserve inequalities, one has

$$a = \lim_{k \to \infty} a \leq \lim_{k \to \infty} 1/k = 0. \quad \Box$$

Remark: The Archimedean Property lies behind the fact that $1/k \to 0$ as $k \to \infty$ in the alternative solution above.

2. [20] Give a counterexample to each of the following assertions.
   (a) Every monotonic sequence in $\mathbb{R}$ converges.
   (b) Every bounded sequence in $\mathbb{R}$ converges.
   (c) A sequence $\{a_k\}_{k \in \mathbb{N}}$ in $\mathbb{R}$ is convergent if the sequence $\{a_k^2\}_{k \in \mathbb{N}}$ is convergent.
   (d) A countable union of closed subsets of $\mathbb{R}$ is closed.

Solution (a): Let $a_k = k$ for every $k \in \mathbb{N}$. Then the sequence $\{a_k\}_{k \in \mathbb{N}}$ is increasing (and therefore monotonic), but does not converge. \hfill \Box

Solution (b): Let $a_k = (-1)^k$ for every $k \in \mathbb{N}$. Then the sequence $\{a_k\}_{k \in \mathbb{N}}$ is bounded ($-1 \leq a_k \leq 1$), but does not converge. \hfill \Box

Solution (c): Let $a_k = (-1)^k$ for every $k \in \mathbb{N}$. Then the sequence $\{a_k^2\}_{k \in \mathbb{N}}$ converges to 1 (because $a_k^2 = (-1)^{2k} = 1$), while the sequence $\{a_k\}_{k \in \mathbb{N}}$ does not converge. \hfill \Box

Solution (d): Let $I_k = \left[\frac{1}{2^k}, 2\right]$ for every $k \in \mathbb{N}$. Then each interval $I_k$ is closed while

$$\bigcup_{k \in \mathbb{N}} I_k = (0, 2] \quad \text{is not closed}. \quad \Box$$
3. [20] Consider the real sequence \( \{b_k\}_{k \in \mathbb{N}} \) given by

\[
b_k = (-1)^k \frac{3k + 4}{k + 1}
\]

for every \( k \in \mathbb{N} = \{0, 1, 2, \ldots\} \).

(a) Give the first three terms of the subsequence \( \{b_{2k+1}\}_{k \in \mathbb{N}} \).

(b) Give the first three terms of the subsequence \( \{b_{3k}\}_{k \in \mathbb{N}} \).

(c) Give \( \limsup_{k \to \infty} b_k \) and \( \liminf_{k \to \infty} b_k \). (No proof is needed here.)

**Solution:** You are given that \( \mathbb{N} = \{0, 1, 2, \ldots\} \), as was done in class and in the notes (but not in the book). Then (a) the first three terms of the subsequence \( \{b_{2k+1}\}_{k \in \mathbb{N}} \) are

\[
b_1 = -\frac{7}{2}, \quad b_3 = -\frac{13}{4}, \quad b_5 = -\frac{19}{6},
\]

while (b) the first three terms of the subsequence \( \{b_{3k}\}_{k \in \mathbb{N}} \) are

\[
b_1 = -\frac{7}{2}, \quad b_3 = -\frac{13}{4}, \quad b_9 = -\frac{31}{10}.
\]

Because \( b_{2k+1} < 0 \) while

\[
\lim_{k \to \infty} b_{2k} = \lim_{k \to \infty} \frac{6k + 4}{2k + 1} = 3,
\]

and because \( b_{2k} > 0 \) while

\[
\lim_{k \to \infty} b_{2k+1} = -\lim_{k \to \infty} \frac{6k + 7}{2k + 2} = -3,
\]

(c) one has respectively that

\[
\limsup_{k \to \infty} b_k = 3, \quad \text{and} \quad \liminf_{k \to \infty} b_k = -3.
\]

\[\square\]

4. [20] Let \( \{a_k\}_{k \in \mathbb{N}} \) and \( \{b_k\}_{k \in \mathbb{N}} \) be bounded sequences in \( \mathbb{R} \).

(a) Prove that

\[
\limsup_{k \to \infty} (a_k + b_k) \leq \limsup_{k \to \infty} a_k + \limsup_{k \to \infty} b_k.
\]

(b) Give an example for which equality does not hold above.

**Solution (a):** Let \( c_k = a_k + b_k \) for every \( k \in \mathbb{N} \). Because \( \{a_k\}_{k \in \mathbb{N}}, \{b_k\}_{k \in \mathbb{N}}, \) and \( \{c_k\}_{k \in \mathbb{N}} \) are bounded sequences in \( \mathbb{R} \), we have

\[
\limsup_{k \to \infty} a_k = \lim_{k \to \infty} \overline{a}_k,
\]

\[
\limsup_{k \to \infty} b_k = \lim_{k \to \infty} \overline{b}_k,
\]

\[
\limsup_{k \to \infty} c_k = \lim_{k \to \infty} \overline{c}_k,
\]

where \( \overline{a}_k, \overline{b}_k, \) and \( \overline{c}_k \) are the corresponding supremums.
where \( \{a_k\}_{k \in \mathbb{N}}, \{b_k\}_{k \in \mathbb{N}}, \) and \( \{c_k\}_{k \in \mathbb{N}} \) are the bounded, non-increasing (and therefore convergent) sequences in \( \mathbb{R} \) whose terms are defined for every \( k \in \mathbb{N} \) by
\[
\begin{align*}
\overline{a}_k &= \sup\{a_l : l \geq k\}, \\
\overline{b}_k &= \sup\{b_l : l \geq k\}, \\
\overline{c}_k &= \sup\{c_l : l \geq k\}.
\end{align*}
\]
For every \( k \in \mathbb{N} \) we have
\[
a_l \leq \overline{a}_k \quad \text{and} \quad b_l \leq \overline{b}_k \quad \text{for every } l \geq k,
\]
whereby
\[
c_l = a_l + b_l \leq \overline{a}_k + \overline{b}_k \quad \text{for every } l \geq k.
\]
By taking the sup over those \( l \) with \( l \geq k \) above, we see that
\[
\overline{c}_k = \sup\{c_l : l \geq k\} \leq \overline{a}_k + \overline{b}_k.
\]
Then by the properties of limits
\[
\limsup_{k \to \infty} (a_k + b_k) = \limsup_{k \to \infty} c_k = \lim_{k \to \infty} \overline{c}_k 
\leq \lim_{k \to \infty} (\overline{a}_k + \overline{b}_k) 
= \lim_{k \to \infty} \overline{a}_k + \lim_{k \to \infty} \overline{b}_k 
= \limsup_{k \to \infty} a_k + \limsup_{k \to \infty} b_k. \quad \Box
\]

**Solution (b):** Let \( a_k = (-1)^k \) and \( b_k = (-1)^{k+1} \) for every \( k \in \mathbb{N} \). Clearly
\[
\limsup_{k \to \infty} a_k = \lim_{k \to \infty} a_{2k} = 1,
\]
\[
\limsup_{k \to \infty} b_k = \lim_{k \to \infty} b_{2k+1} = 1,
\]
while (because \( a_k + b_k = 0 \) for every \( k \in \mathbb{N} \))
\[
\limsup_{k \to \infty} (a_k + b_k) = \lim_{k \to \infty} (a_k + b_k) = 0.
\]
Therefore
\[
\limsup_{k \to \infty} (a_k + b_k) = 0 < 2 = \limsup_{k \to \infty} a_k + \limsup_{k \to \infty} b_k. \quad \Box
\]
5. [20] Let $A$ and $B$ be subsets of $\mathbb{R}$.
(a) Prove that $(A \cap B)^c \subseteq A^c \cap B^c$.
(b) Give an example for which equality does not hold above.

Solution (a): Let $x \in (A \cap B)^c$. By the definition of closure, there exists a sequence $\{x_k\}_{k \in \mathbb{N}}$ contained in $A \cap B$ such that $x_k \to x$ as $k \to \infty$. But the sequence $\{x_k\}_{k \in \mathbb{N}}$ is therefore contained in both $A$ and $B$ while $x_k \to x$ as $k \to \infty$. By the definition of closure, it follows that $x \in A^c$ and $x \in B^c$, whereby $x \in A^c \cap B^c$.

Solution (b): A simple example is $A = (0,1)$ and $B = (1,2)$. Then $(A \cap B)^c = \emptyset = \emptyset$ (because $A \cap B = \emptyset$), while $A^c \cap B^c = [0,1] \cap [1,2] = [1,1] = \{1\}$ (because $A^c = [0,1]$ and $B^c = [1,2]$). Hence, $(A \cap B)^c = \emptyset \neq \{1\} = A^c \cap B^c$.

A more dramatic example is $A = \mathbb{Q}$ and $B = \{\sqrt{2} + q : q \in \mathbb{Q}\}$. Notice that $A \cap B = \emptyset$ because $\sqrt{2}$ is irrational. Notice too that $A^c = \mathbb{R}$ and $B^c = \mathbb{R}$ — i.e. that $A$ and $B$ are each dense in $\mathbb{R}$. Then $(A \cap B)^c = \emptyset^c = \emptyset$, while $A^c \cap B^c = \mathbb{R} \cap \mathbb{R} = \mathbb{R}$. Hence, $(A \cap B)^c = \emptyset \neq \mathbb{R} = A^c \cap B^c$.

6. [10] Let $\{b_k\}_{k \in \mathbb{N}}$ be a sequence in $\mathbb{R}$ and let $A$ be a subset of $\mathbb{R}$. Write the negations of the following assertions.
(a) “For every $m \in \mathbb{R}$ one has $b_j > m$ frequently as $j \to \infty$.”
(b) “Every sequence in $A$ has a subsequence that converges to a limit in $A$.”

Solution (a): “There exists $m \in \mathbb{R}$ such that $b_j \leq m$ ultimately as $j \to \infty$.”

Solution (b): “There is a sequence in $A$ such that every subsequence of it will diverge or will converge to a limit outside $A$.”

Remark: The solution to (b) becomes clearer if you rephrase the original assertion “For every sequence in $A$ there exists a subsequence that will converge and will have its limit in $A$.” The negation of “will converge and will have its limit in $A$” is “will diverge or will converge to a limit outside $A$.”