Final Exam Solutions: MATH 410
Saturday, 16 December 2006

1. [30] State whether each of the following statements is true or false. Give a proof when true and a counterexample when false.

(a) A sequence \( \{a_k\}_{k \in \mathbb{N}} \) in \( \mathbb{R} \) is convergent if the sequence \( \{a_k^2\}_{k \in \mathbb{N}} \) is convergent.

(b) If \( f : \mathbb{R} \rightarrow \mathbb{R} \) is differentiable and increasing over \( \mathbb{R} \) then \( f' > 0 \) over \( \mathbb{R} \).

(c) A function \( f : [a, b] \rightarrow \mathbb{R} \) is Riemann integrable over \( [a, b] \) if the function \( f^2 \) is is Riemann integrable over \( [a, b] \).

Solution (a): This is false. A simple counterexample is given by \( a_k = (-1)^k \) for every \( k \in \mathbb{N} \). Then the sequence \( \{a_k^2\}_{k \in \mathbb{N}} \) converges to 1 (because \( a_k^2 = (-1)^{2k} = 1 \)), while the sequence \( \{a_k\}_{k \in \mathbb{N}} \) does not converge. \( \square \)

Solution (b): This is also false. A simple counterexample is \( f : \mathbb{R} \rightarrow \mathbb{R} \) given by \( f(x) = x^3 \). This function is clearly increasing and differentiable over \( \mathbb{R} \) with \( f'(x) = 3x^2 \). Hence, \( f'(0) = 0 \), which is not positive.

Solution (c): This is also false. A simple counterexample is \( f : [a, b] \rightarrow \mathbb{R} \) given by

\[
  f(x) = \begin{cases} 
    1 & \text{if } x \in \mathbb{Q}, \\
    -1 & \text{otherwise}.
  \end{cases}
\]

The function \( f^2 \) is is Riemann integrable over \([a, b]\) because \( f^2(x) = (f(x))^2 = 1 \) for every \( x \in [a, b] \), while \( f \) is not Riemann integrable over \([a, b]\) because \( \overline{L}(f) = -1 < 1 = \underline{U}(f) \).

Indeed, for every partition \( P \) of \([a, b]\) one has \( L(f, P) = -1 \) and \( U(f, P) = 1 \).

2. [20] Let \( \{a_k\}_{k \in \mathbb{N}} \) and \( \{b_k\}_{k \in \mathbb{N}} \) be bounded sequences in \( \mathbb{R} \).

(a) Prove that \( \lim \sup_{k \to \infty} (a_k + b_k) \leq \lim \sup_{k \to \infty} a_k + \lim \sup_{k \to \infty} b_k \).

(b) Give an example for which equality does not hold above.
Solution (a): Let \( c_k = a_k + b_k \) for every \( k \in \mathbb{N} \). For every \( k \in \mathbb{N} \) we define
\[
\overline{a}_k = \sup \{ a_l : l \geq k \}, \\
\overline{b}_k = \sup \{ b_l : l \geq k \}, \\
\overline{c}_k = \sup \{ c_l : l \geq k \}.
\]
Because \( \{a_k\}_{k \in \mathbb{N}}, \{b_k\}_{k \in \mathbb{N}}, \) and \( \{c_k\}_{k \in \mathbb{N}} \) are bounded above, for every \( k \in \mathbb{N} \) we have
\[
\overline{a}_k < \infty, \quad \overline{b}_k < \infty, \quad \overline{c}_k < \infty.
\]
Therefore \( \{\overline{a}_k\}_{k \in \mathbb{N}}, \{\overline{b}_k\}_{k \in \mathbb{N}}, \) and \( \{\overline{c}_k\}_{k \in \mathbb{N}} \) are nonincreasing sequences in \( \mathbb{R} \). Moreover, because \( \{a_k\}_{k \in \mathbb{N}}, \{b_k\}_{k \in \mathbb{N}}, \) and \( \{c_k\}_{k \in \mathbb{N}} \) are bounded below, the sequences \( \{\overline{a}_k\}_{k \in \mathbb{N}}, \{\overline{b}_k\}_{k \in \mathbb{N}}, \) and \( \{\overline{c}_k\}_{k \in \mathbb{N}} \) are also bounded below. They therefore converge by Montonic Sequence Convergence Theorem. By the definition of \( \limsup \) we have
\[
\limsup_{k \to \infty} a_k = \lim_{k \to \infty} \overline{a}_k, \\
\limsup_{k \to \infty} b_k = \lim_{k \to \infty} \overline{b}_k, \\
\limsup_{k \to \infty} c_k = \lim_{k \to \infty} \overline{c}_k.
\]
The key step is to prove that \( \overline{c}_k \leq \overline{a}_k + \overline{b}_k \) for every \( k \in \mathbb{N} \). Because for every \( k \in \mathbb{N} \) we have
\[
a_l \leq \overline{a}_k, \quad \text{and} \quad b_l \leq \overline{b}_k, \quad \text{for every} \ l \geq k,
\]
it follows that for every \( k \in \mathbb{N} \) we have
\[
c_l = a_l + b_l \leq \overline{a}_k + \overline{b}_k \quad \text{for every} \ l \geq k.
\]
Hence,
\[
\overline{c}_k = \sup \{ c_l : l \geq k \} \leq \overline{a}_k + \overline{b}_k.
\]
Then by the properties of limits
\[
\limsup_{k \to \infty} (a_k + b_k) = \limsup_{k \to \infty} c_k
\]
\[
= \lim_{k \to \infty} \overline{c}_k
\]
\[
\leq \lim_{k \to \infty} (\overline{a}_k + \overline{b}_k)
\]
\[
= \lim_{k \to \infty} \overline{a}_k + \lim_{k \to \infty} \overline{b}_k
\]
\[
= \limsup_{k \to \infty} a_k + \limsup_{k \to \infty} b_k. \quad \square
\]
Solution (b): Let \( a_k = (-1)^k \) and \( b_k = (-1)^{k+1} \) for every \( k \in \mathbb{N} \). Clearly
\[
\limsup_{k \to \infty} a_k = \lim_{k \to \infty} a_{2k} = 1, \\
\limsup_{k \to \infty} b_k = \lim_{k \to \infty} b_{2k+1} = 1,
\]
while (because \( a_k + b_k = 0 \) for every \( k \in \mathbb{N} \))
\[
\limsup_{k \to \infty} (a_k + b_k) = \lim_{k \to \infty} (a_k + b_k) = 0.
\]
Therefore
\[
\limsup_{k \to \infty} (a_k + b_k) = 0 < 2 = \limsup_{k \to \infty} a_k + \limsup_{k \to \infty} b_k. \quad \square
\]

3. [20] Determine all \( a \in \mathbb{R} \) for which the following formal infinite series converge. Give your reasoning.

(a) \( \sum_{n=2}^{\infty} \frac{1}{\log(n)} a^n \)

Solution: The series converges for \( a \in [-1, 1) \) and diverges otherwise.

The cases \( |a| < 1 \) and \( |a| > 1 \) are best handled by the Ratio Test. Let \( b_n = a^n / \log(n) \). Because
\[
\lim_{n \to \infty} \left| \frac{b_{n+1}}{b_n} \right| = \lim_{n \to \infty} \frac{\log(n+1)}{\log(n)} |a| = |a|,
\]
the Ratio Test then implies that this series converges absolutely for \( |a| < 1 \) and diverges for \( |a| > 1 \).

The case \( a = -1 \) is best handled by the Alternating Series Test. Indeed, because the sequence
\[
\left\{ \frac{1}{\log(n)} \right\}_{n=2}^{\infty}
\]
is decreasing and positive.

and because
\[
\lim_{n \to \infty} \frac{1}{\log(n)} = 0,
\]
the Alternating Series Test shows that
\[
\sum_{n=2}^{\infty} (-1)^n \frac{1}{\log(n)} \text{ converges}.
\]
The case $a = 1$ is best handled by Limit Comparison Test, say with the harmonic series. Indeed, because
\[
\lim_{n \to \infty} \frac{\log(n)}{n} = 0,
\]
and because the harmonic series
\[
\sum_{n=2}^{\infty} \frac{1}{n} \text{ diverges},
\]
the Limit Comparison Test shows that
\[
\sum_{n=2}^{\infty} \frac{1}{\log(n)} \text{ diverges}.
\]
Alternatively, one could treat this case with the Direct Comparison Test, the Integral Test, or the Cauchy $2^k$ Test.

\[\]

(b) $\sum_{k=1}^{\infty} \left( \frac{k}{k^5 + 1} \right)^a$

**Solution:** The series converges for $a \in \left( \frac{1}{4}, \infty \right)$ and diverges otherwise. Because
\[
\frac{k}{k^5 + 1} \sim \frac{1}{k^4} \quad \text{as } k \to \infty,
\]
one sees that the original series should be compared with the $p$-series
\[
\sum_{k=1}^{\infty} \frac{1}{k^{4a}}.
\]
This is best handled by Limit Comparison Test. Indeed, because for every $a \in \mathbb{R}$ one has
\[
\lim_{k \to \infty} \frac{\left( \frac{k}{k^5 + 1} \right)^a}{\frac{1}{k^{4a}}} = \lim_{k \to \infty} \left( \frac{k^5}{k^5 + 1} \right)^a = 1,
\]
the Limit Comparison Test then implies that
\[
\sum_{k=1}^{\infty} \left( \frac{k}{k^5 + 1} \right)^a \text{ converges } \iff \sum_{k=1}^{\infty} \frac{1}{k^{4a}} \text{ converges}.
\]
Because $p = 4a$ for the $p$-series, that series converges for $a \in \left( \frac{1}{4}, \infty \right)$ and diverges otherwise. The same is therefore true for the original series. □
4. [20] Let \( f : (a, b) \rightarrow \mathbb{R} \) be uniformly continuous over \( (a, b) \). Let \( \{x_k\}_{k \in \mathbb{N}} \) be a Cauchy sequence contained in \( (a, b) \). Show that \( \{f(x_k)\}_{k \in \mathbb{N}} \) is a Cauchy sequence.

**Solution:** Let \( \varepsilon > 0 \). Because \( f : (a, b) \rightarrow \mathbb{R} \) is uniformly continuous over \( (a, b) \), there exists a \( \delta > 0 \) such that for all points \( x, y \in (a, b) \) one has
\[
|x - y| < \delta \quad \implies \quad |f(x) - f(y)| < \varepsilon .
\]
Because \( \{x_k\}_{k \in \mathbb{N}} \) is a Cauchy sequence, there exists an \( N \in \mathbb{N} \) such that for every \( k, l \in \mathbb{N} \) one has
\[
k, l > N \quad \implies \quad |x_k - x_l| < \delta .
\]
Hence, for every \( k, l \in \mathbb{N} \) one has
\[
k, l > N \quad \implies \quad |x_k - x_l| < \delta \quad \implies \quad |f(x_k) - f(x_l)| < \varepsilon .
\]
Therefore \( \{f(x_k)\}_{k \in \mathbb{N}} \) is a Cauchy sequence.

5. [10] Let \( \{x_n\}_{n=1}^{\infty} \) be a sequence in \( (a, b) \). Give negations of each of the following assertions.

(a) For every \( \varepsilon > 0 \) there exists an \( n_\varepsilon \in \mathbb{N} \) such that
\[
m, n > n_\varepsilon \quad \implies \quad |x_m - x_n| < \varepsilon .
\]

**Solution:** There exists \( \varepsilon > 0 \) such that for every \( l \in \mathbb{N} \) there exists \( m, n \in \mathbb{N} \) such that
\[
m, n > l \quad \text{and} \quad |x_m - x_n| \geq \varepsilon .
\]

(b) There exists a \( c \in \mathbb{R} \) such that no subsequence of \( \{x_n\}_{n=1}^{\infty} \) converges to \( c \).

**Solution:** For every \( c \in \mathbb{R} \) there exists a subsequence of \( \{x_n\}_{n=1}^{\infty} \) that converges to \( c \).

6. [20] Let \( f : (a, b) \rightarrow \mathbb{R} \) be differentiable at a point \( c \in (a, b) \) with \( f'(c) < 0 \). Show that there exists a \( \delta > 0 \) such that
\[
x \in (c - \delta, c) \subset (a, b) \quad \implies \quad f(x) > f(c) ,
\]
\[
x \in (c, c + \delta) \subset (a, b) \quad \implies \quad f(c) > f(x) ,
\]

**Remark:** It is very incorrect to assert that \( f \) is decreasing in an interval containing \( c \).
**Solution:** Because $f$ is differentiable at $c$, we have

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c).$$

Because $f'(c) < 0$ there exists $\delta > 0$ such that $(c - \delta, c + \delta) \subset (a, b)$ and

$$0 < |x - c| < \delta \implies \left| \frac{f(x) - f(c)}{x - c} - f'(c) \right| < -f'(c)$$

$$\implies \frac{f(x) - f(c)}{x - c} < 0.$$

Hence,

$$x \in (c - \delta, c) \implies f(x) - f(c) = \frac{f(x) - f(c)}{x - c} (x - c) > 0$$

$$\implies f(x) > f(c),$$

$$x \in (c, c + \delta) \implies f(x) - f(c) = \frac{f(x) - f(c)}{x - c} (x - c) < 0$$

$$\implies f(x) < f(c).$$

\[\square\]

7. [20] Let $f(x) = \sinh(x) \equiv \frac{1}{2}(e^x - e^{-x})$ for every $x \in \mathbb{R}$. Then for every $k \in \mathbb{N}$ and every $x \in \mathbb{R}$ one has

$$f^{(2k)}(x) = \sinh(x), \quad f^{(2k+1)}(x) = \cosh(x).$$

Show that

$$\sinh(x) = \sum_{k=0}^{\infty} \frac{1}{(2k + 1)!} x^{2k+1} \quad \text{for every } x \in \mathbb{R}.$$  

**Solution:** Because $f(x) = \sinh(x) \equiv \frac{1}{2}(e^x - e^{-x})$, we have $\cosh(x) = f'(x) = \frac{1}{2}(e^x + e^{-x})$. It follows that

$$f^{(2k)}(0) = \sinh(0) = 0, \quad f^{(2k+1)}(0) = \cosh(0) = 1.$$

The series is therefore just the formal Taylor series for $f$ centered at 0. Moreover, we see that the $n^{th}$ partial sum can be expressed as a Taylor polynomial approximation in two ways:

$$\sum_{k=0}^{n} \frac{1}{(2k + 1)!} x^{2k+1} = T_n^{(2n+1)} \sinh(x) = T_n^{(2n+2)} \sinh(x).$$
If we use the last expression, the Lagrange Remainder Theorem then states that for every nonzero $x \in \mathbb{R}$

\[
\sinh(x) = T_0^{(2n+2)} \sinh(x) + \frac{1}{(2n+3)!} \cosh(p) x^{2n+3},
\]

for some $p$ between 0 and $x$. Because $\cosh$ is an even function that is increasing over $[0, \infty)$, for every $p$ between 0 and $x$ one has $\cosh(p) < \cosh(x)$. Hence, for every $x \in \mathbb{R}$

\[
\left| \sinh(x) - \sum_{k=0}^{n} \frac{1}{(2k+1)!} x^{2k+1} \right| \leq \frac{1}{(2n+3)!} \cosh(x) |x|^{2n+3}.
\]

Because for every $x \in \mathbb{R}$

\[
\lim_{n \to \infty} \frac{1}{(2n+3)!} \cosh(x) |x|^{2n+3} = 0,
\]

the sequence of partial sums therefore converges to $\sinh(x)$. □

**Remark:** An alternative approach is to first show that

\[
e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \quad \text{for every } x \in \mathbb{R}.
\]

and then use the fact $f(x) = \sinh(x) = \frac{1}{2}(e^x - e^{-x})$ to derive the series for $\sinh$. The first step uses the Lagrange Remainder Theorem and is given in the notes while second goes like

\[
\sinh(x) = \frac{1}{2}(e^x - e^{-x})
\]

\[
= \frac{1}{2} \left( \sum_{n=0}^{\infty} \frac{1}{n!} x^n - \sum_{n=0}^{\infty} \frac{1}{n!} (-x)^n \right)
\]

\[
= \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1 - (-1)^n}{2} x^n
\]

\[
= \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} x^{2k+1}.
\]

8. [20] Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable. Show that for every $\epsilon > 0$ there exists a partition $P$ of $[a, b]$ such that

\[
0 \leq U(f, P) - L(f, P) < \epsilon,
\]

where $L(f, P)$ and $U(f, P)$ are the lower and upper Darboux sums associated with $f$ and $P$. 

Solution: Let $\epsilon > 0$. Because
\begin{align*}
\overline{L}(f) &= \sup\{ L(f, P) : P \text{ is a partition of } [a, b] \}, \\
\underline{U}(f) &= \inf\{ U(f, P) : P \text{ is a partition of } [a, b] \},
\end{align*}
there exists partitions $P^L$ and $P^U$ of $[a, b]$ such that
\begin{align*}
\overline{L}(f) - \frac{\epsilon}{2} < L(f, P^L) \leq \overline{L}(f), \\
\underline{U}(f) \leq U(f, P^U) < \underline{U}(f) + \frac{\epsilon}{2}.
\end{align*}

Let $P^* = P^L \lor P^U$. Then by the Refinement Lemma
\begin{align*}
\overline{L}(f) - \frac{\epsilon}{2} < L(f, P^L) \leq L(f, P^*) \leq \overline{L}(f), \\
\underline{U}(f) \leq U(f, P^*) \leq U(f, P^U) < \underline{U}(f) + \frac{\epsilon}{2}.
\end{align*}

Because $f$ is Riemann integrable, $\overline{L}(f) = \underline{U}(f)$. Hence,
\begin{align*}
0 \leq U(f, P^*) - L(f, P^*) < \left( \underline{U}(f) + \frac{\epsilon}{2} \right) - \left( \overline{L}(f) - \frac{\epsilon}{2} \right) = \epsilon.
\end{align*}
\hfill\Box

9. [20] Let $f : [a, b] \to \mathbb{R}$ be continuous. Prove that there exists $p \in (a, b)$ such that
\begin{align*}
f(p) = \frac{1}{e^b - e^a} \int_a^b f(x)e^x \, dx.
\end{align*}

Solution: Let $g : [a, b] \to \mathbb{R}$ be given by $g(x) = e^x$ for every $x \in [a, b]$. Clearly $g$ is Riemann integrable over $[a, b]$. Because $f : [a, b] \to \mathbb{R}$ is continuous while $g : [a, b] \to \mathbb{R}$ is positive and Riemann integrable, the Integral Mean-Value Theorem implies there exists $p \in (a, b)$ such that
\begin{align*}
\int_a^b f(x)g(x) \, dx = f(p) \int_a^b g(x) \, dx.
\end{align*}

But
\begin{align*}
\int_a^b g(x) \, dx &= \int_a^b e^x \, dx = e^b - e^a > 0,
\end{align*}
so that
\begin{align*}
f(p) &= \frac{1}{e^b - e^a} \int_a^b f(x)e^x \, dx.
\end{align*}
\hfill\Box
10. [20] Prove that every countable set has measure zero.

**Solution:** Let $A \subset \mathbb{R}$ be countable. Let $\epsilon > 0$. Because $A$ is countable there exists a sequence $\{x_k\}_{k \in \mathbb{N}} \subset \mathbb{R}$ such that $A \subset \{x_k\}_{k \in \mathbb{N}}$. Let $r < \frac{1}{2}$. Then

$$A \subset \{x_k\}_{k \in \mathbb{N}} \subset \bigcup_{k \in \mathbb{N}} (x_k - r^{k+2}\epsilon, x_k + r^{k+2}\epsilon) ,$$

while (because $r < \frac{1}{2}$ implies $2r^2/(1-r) < 1$)

$$\sum_{k=0}^{\infty} 2r^{k+2}\epsilon = \frac{2r^2\epsilon}{1-r} < \epsilon .$$

But $\epsilon > 0$ was arbitrary, so $A$ has measure zero. \qed