Second In-Class Exam Solutions
Math 246, Spring 2008, Professor David Levermore

(1) [6] Give the interval of existence for the solution of the initial-value problem

$$\frac{\mathrm{d}^3 z}{\mathrm{d}t^3} + \frac{e^t}{2-t}\frac{\mathrm{d}z}{\mathrm{d}t} = \tan(t), \qquad z(\pi) = z'(\pi) = z''(\pi) = 0.$$

Solution. The coefficient and forcing are both continuous over the interval $(2, \frac{3\pi}{2})$, which contains the initial time $t = \pi$. The coefficient is not defined at $t = 2$ while the forcing is not defined at $t = \frac{3\pi}{2}$. The interval of existence is therefore $(2, \frac{3\pi}{2})$.

(2) [9] Solve the initial-value problem

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + 9y = 3t, \qquad y(0) = 0, \quad y'(0) = 1.$$

Solution. This is a constant coefficient, inhomogeneous, linear equation. Its characteristic polynomial is

$$P(z) = z^2 + 9 = z^2 + 3^2.$$

This has the conjugate pair of roots $\pm i3$, which yields a general solution of the associated homogeneous problem

$$y_H(t) = c_1 \cos(3t) + c_2 \sin(3t).$$

The forcing $3t$ has degree $d = 1$ and characteristic $r + is = 0$, which is a root of $P(z)$ of multiplicity $m = 0$. A particular solution $y_P(t)$ can be found by the method of undetermined coefficients using either direct substitution (as in the book) or KEY identity evaluation (as in the lectures). Alternatively, one can solve the problem with Laplace transform.

KEY Indentity Evaluations. Because $m + d = 1$, you need the KEY identity and its first derivative

$$\mathbf{L}(e^{zt}) = P(z)e^{zt} = (z^2 + 9)e^{zt}, \qquad \mathbf{L}(t\,e^{zt}) = (z^2 + 9)t\,e^{zt} + 2z\,e^{zt}.$$

Evaluate these at $z = 0$ to find $\mathbf{L}(1) = 9$ and $\mathbf{L}(t) = 9t$. Dividing the second of these equations by 3 yields $\mathbf{L}(\frac{1}{3}t) = 3t$, which implies $y_P(t) = \frac{1}{3}t$.

Direct Substitution. Because $m = 0$ and $m + d = 1$, you seek a particular solution of the form

$$y_P(t) = A_0 t + A_1.$$

Because $y_P'(t) = A_0$ and $y_P''(t) = 0$, one sees that

$$\mathbf{L}y_P(t) = y_P''(t) + 9y_P(t) = 9(A_0 t + A_1) = 9A_0 t + 9A_1.$$

Setting $\mathbf{L}y_P(t) = 9A_0 t + 9A_1 = 3t$, we see that $9A_0 = 3$ and $9A_1 = 0$, whereby $A_0 = \frac{1}{3}$ and $A_1 = 0$. Hence, $y_P(t) = \frac{1}{3}t$.

Imposing the Initial Conditions. By either approach one finds $y_P(t) = \frac{1}{3}t$, which yields the general solution

$$y(t) = c_1 \cos(3t) + c_2 \sin(3t) + \tfrac{1}{3}t .$$

Because

$$y'(t) = -3c_1 \sin(3t) + 3c_2 \cos(3t) + \tfrac{1}{3} ,$$

when the initial conditions are imposed, one finds that

$$y(0) = c_1 = 0 , \qquad y'(0) = 3c_2 + \tfrac{1}{3} = 1 .$$

These are solved to find $c_1 = 0$ and $c_2 = \frac{2}{9}$. The solution of the initial-value problem is therefore

$$y(t) = \tfrac{2}{9} \sin(3t) + \tfrac{1}{3}t .$$

Laplace Transform. The Laplace transform of the initial-value problem is

$$\mathcal{L}[y''](s) + 9\mathcal{L}[y](s) = 3\mathcal{L}[t](s) = 3\frac{1}{s^2} ,$$

where

$$\mathcal{L}[y](s) = Y(s) ,$$
$$\mathcal{L}[y''](s) = s^2 Y(s) - sy(0) - y'(0) = s^2 Y(s) - 1 .$$

Hence,

$$(s^2 + 9)Y(s) = 1 + \frac{3}{s^2} ,$$

whereby

$$Y(s) = \frac{1}{s^2 + 9} + \frac{3}{s^2(s^2 + 9)} .$$

A partial fraction decomposition yields

$$\frac{3}{s^2(s^2 + 9)} = \frac{\frac{1}{3}}{s^2} - \frac{\frac{1}{3}}{s^2 + 9} ,$$

whereby

$$Y(s) = \frac{\frac{2}{3}}{s^2 + 9} + \frac{\frac{1}{3}}{s^2} .$$

You see from the table on the last page that

$$\mathcal{L}[\sin(3t)](s) = \frac{3}{s^2 + 9} , \quad \text{and} \quad \mathcal{L}[t] = \frac{1}{s^2} ,$$

whereby

$$Y(s) = \tfrac{2}{9}\mathcal{L}[\sin(3t)](s) + \tfrac{1}{3}\mathcal{L}[t](s) = \mathcal{L}\left[\tfrac{2}{9}\sin(3t) + \tfrac{1}{3}t\right](s) .$$

The solution of the initial-value problem is therefore

$$y(t) = \mathcal{L}^{-1}[Y](t) = \tfrac{2}{9} \sin(3t) + \tfrac{1}{3}t .$$

(3) [9] Let $\mathbf{D} = \dfrac{\mathrm{d}}{\mathrm{d}t}$. Give a general real solution of the equation

$$\mathbf{D}^2 y - 2\mathbf{D}y + 5y = 10\cos(t).$$

Solution. This is a constant coefficient, inhomogeneous, linear equation. Its characteristic polynomial is

$$P(z) = z^2 - 2z + 5 = (z-1)^2 + 4 = (z-1)^2 + 2^2.$$

This has the conjugate pair of roots $1 \pm i2$, which yields a general solution of the associated homogeneous problem

$$y_H(t) = c_1 e^t \cos(2t) + c_2 e^t \sin(2t).$$

The forcing $10\cos(t)$ has degree $d = 0$ and characteristic $r + is = i$, which is a root of $P(z)$ of multiplicity $m = 0$. A particular solution $y_P(t)$ can be found by the method of undetermined coefficients using either direct substitution (as in the book) or KEY identity evaluation (as in the lectures).

KEY Indentity Evaluations. Because $m + d = 0$, you only need to evaluate the KEY identity at $z = i$, which yields

$$\mathbf{L}(e^{it}) = P(i)e^{it} = (i^2 - 2i + 5)e^{it} = (4 - i2)e^{it}.$$

Because the forcing has the form $10\cos(t) = 10\operatorname{Re}(e^{it})$, we write

$$\mathbf{L}\left(\frac{10e^{it}}{4 - i2}\right) = 10e^{it},$$

which implies that

$$y_P(t) = \operatorname{Re}\left(\frac{5e^{it}}{2 - i}\right) = \operatorname{Re}\left(\frac{5e^{it}}{2 - i}\frac{2 + i}{2 + i}\right) = \operatorname{Re}\left(\frac{5(2 + i)e^{it}}{2^2 + 1^2}\right)$$

$$= \operatorname{Re}\big((2 + i)e^{it}\big) = 2\cos(t) - \sin(t).$$

A general solution of the equation is therefore

$$y(t) = c_1 e^t \cos(2t) + c_2 e^t \sin(2t) + 2\cos(t) - \sin(t).$$

Direct Substitution. Because $m = d = 0$, you seek a particular solution of the form

$$y_P(t) = A\cos(t) + B\sin(t).$$

Because

$$y_P'(t) = -A\sin(t) + B\cos(t), \qquad y_P''(t) = -A\cos(t) - B\sin(t),$$

one sees that

$$\mathbf{L}y_P(t) = y_P''(t) - 2y_P'(t) + 5y_P(t) = (4A - 2B)\cos(t) + (2A + 4B)\sin(t).$$

Setting $\mathbf{L}y_P(t) = 10\cos(t)$, we see that

$$4A - 2B = 10, \qquad 2A + 4B = 0,$$

whereby $A = 2$ and $B = -1$. Hence, $y_P(t) = 2\cos(t) - \sin(t)$. A general solution of the equation is therefore

$$y(t) = c_1 e^t \cos(2t) + c_2 e^t \sin(2t) + 2\cos(t) - \sin(t).$$

(4) [9] What answer will be produced by the following MATLAB commands?

>> ode = 'D2y + 4*Dy + 8*y = 16*exp(–2*t)';
>> dsolve(ode, 't')

ans =

Solution. The commands ask MATLAB to give a general solution of the equation

$$\mathbf{D}^2 y + 4\mathbf{D}y + 8y = 16e^{-2t}, \qquad \text{where} \quad \mathbf{D} = \frac{d}{dt}.$$

MATLAB will produce the answer

exp(–2t)*sin(2*t)*C2 + exp(–2t)*cos(2*t)*C1 + 4*exp(–2*t)

This can be seen as follows. This is a constant coefficient, inhomogeneous, linear equation. The characteristic polynomial is

$$P(z) = z^2 + 4z + 8 = (z+2)^2 + 4 = (z+2)^2 + 2^2.$$

Its roots are the conjugate pair $-2 \pm i2$. A general solution of the associated homogeneous problem is

$$y_H(t) = c_1 e^{-2t} \cos(2t) + c_2 e^{-2t} \sin(2t).$$

The forcing $16e^{-2t}$ has degree $d = 0$ and characteristic $r + is = -2$, which is a root of $P(z)$ of multiplicity $m = 0$. A particular solution $y_P(t)$ can be found by the method of undetermined coefficients using either direct substitution (as in the book) or KEY identity evaluation (as in the lectures).

KEY Indentity Evaluations. Because $m + d = 0$, you only need to evaluate the KEY identity at the characteristic $z = -2$, which yields

$$\mathbf{L}(e^{-2t}) = P(-2)e^{-2t} = \big((-2)^2 + 4(-2) + 8\big)e^{-2t} = 4e^{-2t}.$$

Multiplying this by 4 yields $\mathbf{L}(4e^{-2t}) = 16e^{-2t}$, which implies $y_P(t) = 4e^{-2t}$.

Direct Substitution. Because $m = d = 0$ and the characteristic is 0, you seek a particular solution of the form

$$y_P(t) = Ae^{-2t}.$$

Because

$$y_P'(t) = -2Ae^{-2t}, \qquad y_P''(t) = 4Ae^{-2t},$$

one sees that

$$\mathbf{L}y_P(t) = y_P''(t) + 4y_P'(t) + 8y_P(t)$$
$$= [4Ae^{-2t}] + 4[-2Ae^{-2t}] + 8[Ae^{-2t}] = 4Ae^{-2t}.$$

Setting $\mathbf{L}y_P(t) = 4Ae^{-2t} = 16e^{-2t}$, we see that $A = 4$. Hence, $y_P(t) = 4e^{-2t}$.

By either approach you find $y_P(t) = 4e^{-2t}$. A general solution is therefore

$$y(t) = c_1 e^{-2t} \cos(2t) + c_2 e^{-2t} \sin(2t) + 4e^{-2t}.$$

Up to notational differences, this is the answer that MATLAB produces.

(5) [8] Let **L** be a linear ordinary differential operator with constant coefficients. Suppose that all the roots of its characteristic polynomial (listed with their multiplicities) are $-2 + i3$, $-2 + i3$, $-2 - i3$, $-2 - i3$, $i5$, $-i5$, -7, -7, 0, 0, 0.

(a) Give the order of **L**.

Solution. There are 11 roots listed above, so the degree of the characteristic polynomial is 11, whereby the order of **L** is 11.

(b) Give a general real solution of the homogeneous equation $\mathbf{L}y = 0$.

Solution. A general solution is

$$y(t) = c_1 e^{-2t}\cos(3t) + c_2 e^{-2t}\sin(3t) + c_3 t\, e^{-2t}\cos(3t) + c_4 t\, e^{-2t}\sin(3t)$$
$$= c_5\cos(5t) + c_6\sin(5t) + c_7 e^{-7t} + c_8 t\, e^{-7t} + c_9 + c_{10}t + c_{11}t^2 .$$

The reasoning is as follows:
- the double conjugate pair $-2 \pm i3$ yields
$$e^{-2t}\cos(3t), \quad e^{-2t}\sin(3t), \quad t\, e^{-2t}\cos(3t), \quad \text{and} \quad t\, e^{-2t}\sin(3t) ;$$
- the single conjugate pair $\pm i5$ yields $\cos(5t)$ and $\sin(5t)$;
- the double real root -7 yields e^{-7t} and $t\, e^{-7t}$;
- the triple real root 0 yields 1, t, and t^2.

(6) [9] The functions x^2 and x^3 are solutions of the homogeneous equation

$$x^2 \frac{d^2 y}{dx^2} - 4x \frac{dy}{dx} + 6y = 0 \qquad \text{over } x > 0 .$$

(You do not have to check that this is true!)

(a) Compute their Wronskian.

Solution. The Wronskian is

$$W[x^2, x^3](x) = \det \begin{pmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{pmatrix} = 3x^4 - 2x^4 = x^4 .$$

(b) Give a general solution of the equation

$$x^2 \frac{d^2 y}{dx^2} - 4x \frac{dy}{dx} + 6y = 4x^4 e^{2x} \qquad \text{over } x > 0 .$$

You may express the solution in terms of definite integrals.

Solution. Because $W[x^2, x^3](x) = x^4 > 0$ over $x > 0$, the functions x^2 and x^3 are linearly independent. A general solution of the associated homogeneous problem is

$$y_H(x) = c_1 x^2 + c_2 x^3 .$$

Because this problem does not have constant coefficients, you must use the method of variation of parameters to find a particular solution $y_P(x)$. First, divide by x^2 to bring the equation into its normal form

$$\frac{d^2 y}{dx^2} - \frac{4}{x}\frac{dy}{dx} + \frac{6}{x^2} y = 4x^2 e^{2x} \qquad \text{over } x > 0 .$$

Seek a solution in the form

$$y = u_1(x)x^2 + u_2(x)x^3 \,.$$

where $u_1'(x)$ and $u_2'(x)$ satisfy

$$u_1'(x)x^2 + u_2'(x)x^3 = 0 \,,$$
$$u_1'(x)2x + u_2'(x)3x^2 = 4x^2 e^{2x} \,.$$

Solve this system to find

$$u_1'(x) = -4x\, e^{2x} \,, \qquad u_2'(x) = 4e^{2x} \,.$$

Integrate these equations to find

$$u_1(x) = c_1 + (1 - 2x)e^{2x} \,, \qquad u_2(x) = c_2 + 2e^{2x} \,.$$

A general solution is therefore

$$y = c_1 x^2 + c_2 x^3 + (1 - 2x)e^{2x}x^2 + 2e^{2x}x^3 = c_1 x^2 + c_2 x^3 + x^2 e^{2x} \,.$$

(7) [8] The vertical displacement of a mass on a spring is given by

$$h(t) = 3\cos(8t) - 4\sin(8t) \,.$$

Express this in the form $h(t) = A\cos(\omega t - \delta)$ with $A > 0$ and $0 \le \delta < 2\pi$, identifying the period, amplitude, and phase of the oscillation. (The phase may be expressed in terms of an inverse trig function.)

Solution. By compairing

$$A\cos(\omega t - \delta) = A\cos(\delta)\cos(\omega t) + A\sin(\delta)\sin(\omega t)$$

with $h(t) = 3\cos(8t) - 4\sin(8t)$, we see that $\omega = 8$ and that

$$A\cos(\delta) = 3 \,, \qquad A\sin(\delta) = -4 \,.$$

Therefore the period T is given by

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{8} = \frac{\pi}{4} \,,$$

while the amplitude A is given by

$$A = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5 \,.$$

Because $\cos(\delta) = \frac{3}{5} > 0$ while $\sin(\delta) = -\frac{4}{5} < 0$, the phase δ lies in the fourth quadrant $\left(\frac{3\pi}{2} < \delta < 2\pi\right)$ and satisfies any one of the formulas

$$\sin(\delta) = -\tfrac{4}{5} \,, \qquad \cos(\delta) = \tfrac{3}{5} \,, \qquad \tan(\delta) = -\tfrac{4}{3} \,.$$

It is therefore given by any one of the following formulas

$$\delta = 2\pi + \sin^{-1}\left(-\tfrac{4}{5}\right) \,, \qquad \delta = 2\pi - \cos^{-1}\left(\tfrac{3}{5}\right) \,, \qquad \delta = 2\pi + \tan^{-1}\left(-\tfrac{4}{3}\right) \,.$$

(8) [9] When a mass of 2 kilograms is hung vertically from a spring, at rest it stretches the spring .2 m. (Gravitational acceleration is $g = 9.8$ m/sec^2.) At $t = 0$ the mass is displaced .1 m above its rest position and is released with a downward initial velocity of .3 m/sec. Assume that the spring force is proportional to displacement, that there is no drag force, and that the mass is driven by an external force of $F_{ext}(t) = 10\cos(\omega t)$ Newtons (1 Newton $= 1$ kg m/sec^2), where up is taken to be positive.

  (a) Formulate an initial-value problem that governs the motion of the mass for $t > 0$. (DO NOT solve this initial-value problem, just write it down!)

    Solution. Let $h(t)$ be the displacement of the mass from its equilibrium (rest) position at time t in centimeters, with upward displacements being positive. The governing initial-value problem then has the form

$$m\frac{d^2 h}{dt^2} + kh = F_{ext}(t)\,, \qquad h(0) = .1\,, \quad h'(0) = -.3\,,$$

    where m is the mass and k is the spring constant. The problem says that $m = 2$ kilograms. The spring constant is obtained by balancing the weight of the mass ($mg = 2 \cdot 9.8$ Newtons) with the force applied by the spring when it is stetched .2 m. This gives $k\,.2 = 2 \cdot 9.8$, or

$$k = \frac{2 \cdot 9.8}{.2} = \frac{2 \cdot 98}{2} = 98 \quad \text{Newtons/m}\,.$$

    Because $F_{ext}(t) = 10\cos(\omega t)$, the governing initial-value problem is therefore

$$2\frac{d^2 h}{dt^2} + 98h = 10\cos(\omega t)\,, \qquad h(0) = .1\,, \quad h'(0) = -.3\,.$$

  (b) What is the natural frequency of this spring?

    Solution. The natural frequency of the spring is given by

$$\omega_o = \sqrt{\frac{k}{m}} = \sqrt{\frac{98}{2}} = \sqrt{49} = 7 \quad 1/\text{sec}\,.$$

  (c) At what value of the driving frequency ω does resonance occur?

    Solution. Resonance occurs when the driving frequency ω equals the natural fequency of the spring ω_o. Given the answer to part (b), resonance occurs when

$$\omega = \omega_o = 7 \quad 1/\text{sec}\,.$$

(9) [8] Compute the Laplace transform of $f(t) = u(t-2)\, e^{3t}$ from its definition. (Here u is the unit step function.)

Solution. The definition of Laplace transform gives

$$\mathcal{L}[f](s) = \lim_{T\to\infty} \int_0^T e^{-st} u(t-2)\, e^{3t}\, \mathrm{d}t = \lim_{T\to\infty} \int_2^T e^{(3-s)t}\, \mathrm{d}t\,.$$

This limit diverges to $+\infty$ for $s \le 3$ because in that case

$$\int_2^T e^{(3-s)t}\, \mathrm{d}t \ge \int_2^T \mathrm{d}t = T - 2\,,$$

which clearly diverges to $+\infty$ as $T \to \infty$.

For $s > 3$ an integration by parts shows that

$$\int_2^T e^{(3-s)t}\, \mathrm{d}t = \left. \frac{e^{(3-s)t}}{3-s} \right|_2^T = \frac{e^{(3-s)T}}{3-s} - \frac{e^{(3-s)2}}{3-s}\,.$$

Hence, for $s > 3$ one has that

$$\mathcal{L}[f](s) = \lim_{T\to\infty} \left(\frac{e^{-(s-3)2}}{s-3} - \frac{e^{-(s-3)T}}{s-3} \right) = \frac{e^{-(s-3)2}}{s-3} - \lim_{T\to\infty} \frac{e^{-(s-3)T}}{s-3}$$

$$= \frac{e^{-(2s-6)}}{s-3}\,.$$

(10) [9] Find the Laplace transform $Y(s)$ of the solution $y(t)$ of the initial-value problem

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + 6\frac{\mathrm{d}y}{\mathrm{d}t} + 25y = f(t)\,, \qquad y(0) = 1\,, \quad y'(0) = -3\,,$$

where

$$f(t) = \begin{cases} \sin(2t) & \text{for } 0 \le t < \pi\,, \\ (t-\pi)^2 & \text{for } t \ge \pi\,. \end{cases}$$

You may refer to the table on the last page. DO NOT take the inverse Laplace transform to find $y(t)$, just solve for $Y(s)$!

Solution. The Laplace transform of the initial-value problem is

$$\mathcal{L}[y''](s) + 6\mathcal{L}[y'](s) + 25\mathcal{L}[y](s) = \mathcal{L}[f](s)\,,$$

where

$$\mathcal{L}[y](s) = Y(s)\,,$$
$$\mathcal{L}[y'](s) = sY(s) - y(0) = sY(s) - 1\,,$$
$$\mathcal{L}[y''](s) = s^2 Y(s) - sy(0) - y'(0) = s^2 Y(s) - s + 3\,.$$

To compute $\mathcal{L}[f](s)$, first write f as

$$f(t) = \bigl(1 - u(t-\pi)\bigr)\sin(2t) + u(t-\pi)(t-\pi)^2$$
$$= \sin(2t) - u(t-\pi)\sin(2t) + u(t-\pi)(t-\pi)^2$$
$$= \sin(2t) - u(t-\pi)\sin\bigl(2(t-\pi)\bigr) + u(t-\pi)(t-2\pi)^2\,.$$

Referring to the table on the last page, item 5 with $c = \pi$, item 3 with $b = 2$, and item 1 with $n = 2$ then show that

$$\mathcal{L}[f](s) = \mathcal{L}[\sin(2t)](s) - \mathcal{L}\big[u(t - 2\pi)\sin\big(2(t - \pi)\big)\big](s) + \mathcal{L}\big[u(t - \pi)(t - \pi)^2\big](s)$$

$$= \mathcal{L}[\sin(2t)](s) - e^{-\pi s}\mathcal{L}[\sin(2t)](s) + e^{-\pi s}\mathcal{L}[t^2](s)$$

$$= \big(1 - e^{-\pi s}\big)\frac{2}{s^2 + 2^2} + e^{-\pi s}\frac{2}{s^3}.$$

The Laplace transform of the initial-value problem then becomes

$$\big(s^2 Y(s) - s + 3\big) + 6\big(sY(s) - 1\big) + 25Y(s) = \big(1 - e^{-\pi s}\big)\frac{2}{s^2 + 4} + e^{-\pi s}\frac{2}{s^3},$$

which becomes

$$(s^2 + 6s + 25)Y(s) - s + 3 - 6 = \big(1 - e^{-\pi s}\big)\frac{2}{s^2 + 4} + e^{-\pi s}\frac{2}{s^3}.$$

Hence, $Y(s)$ is given by

$$Y(s) = \frac{1}{s^2 + 6s + 25}\left(s + 3 + \big(1 - e^{-\pi s}\big)\frac{2}{s^2 + 4} + e^{-\pi s}\frac{2}{s^3}\right).$$

(11) [16] Find the inverse Laplace transforms of the following functions. You may refer to the table on the last page.

(a) $F(s) = \dfrac{8}{s^2 - 2s - 3}$,

Solution. The denominator factors as $(s - 3)(s + 1)$, so the partial fraction decomposition is

$$\frac{8}{s^2 - 2s - 3} = \frac{8}{(s - 3)(s + 1)} = \frac{2}{s - 3} - \frac{2}{s + 1}.$$

Referring to the table on the last page, item 1 with $n = 0$ gives $\mathcal{L}[1](s) = 1/s$. Item 4 with $a = 3$ and $f(t) = 1$, and with $a = -2$ and $f(t) = 1$, then gives

$$\mathcal{L}[e^{3t}](s) = \frac{1}{s - 3}, \qquad \mathcal{L}[e^{-t}](s) = \frac{1}{s + 1},$$

whereby

$$\frac{8}{s^2 - 2s - 3} = 2\mathcal{L}[e^{3t}](s) - 2\mathcal{L}[e^{-t}](s) = \mathcal{L}\big[2e^{3t} - 2e^{-t}\big](s).$$

You therefore conclude that

$$\mathcal{L}^{-1}\left[\frac{8}{s^2 - 2s - 3}\right](t) = 2e^{3t} - 2e^{-t}.$$

(b) $F(s) = \dfrac{4e^{-\pi s}}{s^2 - 6s + 25}$.

Solution. Complete the square in the denominator to get $(s-3)^2 + 4^2$. Referring to the table on the last page, item 3 with $b = 4$ gives

$$\mathcal{L}[\sin(4t)](s) = \frac{4}{s^2 + 4^2} \,.$$

Item 4 with $a = 3$ and $f(t) = \sin(4t)$ then gives

$$\mathcal{L}[e^{3t}\sin(4t)](s) = \frac{4}{(s-3)^2 + 4^2} \,.$$

Item 5 with $c = \pi$ and $f(t) = e^{3t}\sin(4t)$ then gives

$$\mathcal{L}\big[u(t-\pi)e^{3(t-\pi)}\sin\big(4(t-\pi)\big)\big](s) = e^{-\pi s}\frac{4}{(s-3)^2 + 4^2} \,.$$

You therefore conclude that

$$\mathcal{L}^{-1}\bigg[e^{-\pi s}\frac{4}{s^2 - 6s + 25}\bigg](t) = u(t-\pi)e^{3(t-\pi)}\sin(4t - 4\pi)$$

$$= u(t-\pi)e^{3(t-\pi)}\sin(4t)\,.$$

A Short Table of Laplace Transforms

$$\mathcal{L}[t^n](s) = \frac{n!}{s^{n+1}} \qquad\qquad \text{for } s > 0\,.$$

$$\mathcal{L}[\cos(bt)](s) = \frac{s}{s^2 + b^2} \qquad\qquad \text{for } s > 0\,.$$

$$\mathcal{L}[\sin(bt)](s) = \frac{b}{s^2 + b^2} \qquad\qquad \text{for } s > 0\,.$$

$$\mathcal{L}[e^{at}f(t)](s) = F(s-a) \qquad\qquad \text{where } F(s) = \mathcal{L}[f(t)](s)\,.$$

$$\mathcal{L}[u(t-c)f(t-c)](s) = e^{-cs}F(s) \qquad\qquad \text{where } F(s) = \mathcal{L}[f(t)](s)$$
$$\text{and } u \text{ is the unit step function}\,.$$