Third In-Class Exam Solutions  
Math 246, Spring 2008, Professor David Levermore

(1) [10] Consider the matrices

\[ A = \begin{pmatrix} i7 & 2 + i \\ 1 - i & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}. \]

Compute the matrices

(a) \( A^T \)  
Solution. \( A^T = \begin{pmatrix} i7 & 1 - i \\ 2 + i & 4 \end{pmatrix} \)

(b) \( \overline{A} \)  
Solution. \( \overline{A} = \begin{pmatrix} -i7 & 2 - i \\ 1 + i & 4 \end{pmatrix} \)

(c) \( 2A - B \)  
Solution. \( 2A - B = \begin{pmatrix} i14 & 4 + i2 \\ 2 - i2 & 8 \end{pmatrix} - \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} -5 + i14 & 1 + i2 \\ -1 - i2 & 6 \end{pmatrix} \)

(d) \( AB \)  
Solution. \( AB = \begin{pmatrix} i7 & 2 + i \\ 1 - i & 4 \end{pmatrix} \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 6 + i38 & 4 + i23 \\ 17 - i5 & 11 - i4 \end{pmatrix} \)

(e) \( B^{-1} \)  
Solution. Because \( \det(B) = 5 \cdot 2 - 3 \cdot 3 = 10 - 9 = 1 \),

\[ B^{-1} = \frac{1}{\det(B)} \begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix}. \]

(2) [8] Consider the matrix

\[ A = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}. \]

(a) Find all the eigenvalues of \( A \).

Solution. The characteristic polynomial of \( A \) is given by

\[ p(z) = z^2 - \text{tr}(A)z + \det(A) = z^2 - 5z + 4 = (z - 1)(z - 4). \]

The eigenvalues of \( A \) are the roots of this polynomial, which are 1 and 4.

(b) For each eigenvalue of \( A \) find an associated eigenvector.

Solution (using the Cayley-Hamilton method from notes). One has

\[ A - I = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}, \quad A - 4I = \begin{pmatrix} -1 & 2 \\ 1 & -2 \end{pmatrix}. \]

Every nonzero column of \( A - 4I \) has the form

\[ \alpha_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ for some } \alpha_1 \neq 0, \]

any of which is an eigenvector associated with 4. Similarly, every nonzero column of \( A - I \) has the form

\[ \alpha_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} \text{ for some } \alpha_2 \neq 0, \]

any of which is an eigenvector associated with 1.
(3) [6] Transform the equation \( \frac{d^4u}{dt^4} + e^t \frac{d^2u}{dt^2} - 5u = \cos(t) \) into a first-order system of ordinary differential equations.

**Solution:** Because the equation is fourth order, the first order system must have dimension four. The simplest such first order system is

\[
\begin{pmatrix}
\frac{dx_1}{dt} \\
\frac{dx_2}{dt} \\
\frac{dx_3}{dt} \\
\frac{dx_4}{dt}
\end{pmatrix}
= \begin{pmatrix}
x_2 \\
x_3 \\
x_4 \\
\cos(t) + 5x_1 - e^t x_3
\end{pmatrix},
\]

where

\[
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{pmatrix}
= \begin{pmatrix}
u \\
u' \\
u'' \\
u'''
\end{pmatrix}.
\]

(4) [10] Consider the vector-valued functions \( x_1(t) = \left( t^2 + 1 \right) \), \( x_2(t) = \left( t \right) \).

(a) Compute the Wronskian \( W[x_1, x_2](t) \).

**Solution.**

\[
W[x_1, x_2](t) = \det \begin{pmatrix} t^2 + 1 & t \\ t & 1 \end{pmatrix} = t^2 + 1 - t^2 = 1.
\]

(b) Find \( A(t) \) such that \( x_1, x_2 \) is a fundamental set of solutions to the system \( \frac{dx}{dt} = A(t)x \) wherever \( W[x_1, x_2](t) \neq 0 \).

**Solution.** Let \( \Psi(t) = \begin{pmatrix} t^2 + 1 & t \\ t & 1 \end{pmatrix} \). Because \( \frac{\Psi(t)}{dt} = A(t)\Psi(t) \), one has

\[
A(t) = \frac{\Psi(t)}{dt} \Psi(t)^{-1} = \begin{pmatrix} 2t & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} t^2 + 1 & t \\ t & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 2t & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -t \\ -t & t^2 + 1 \end{pmatrix} = \begin{pmatrix} t & 1 - t^2 \\ 1 & -t \end{pmatrix}.
\]

(c) Give a general solution to the system you found in part (b).

**Solution.** Because \( W[x_1, x_2](t) = 1 \neq 0 \), a general solution is

\[
x = c_1 x_1(t) + c_2 x_2(t) = c_1 \begin{pmatrix} t^2 + 1 \\ t \end{pmatrix} + c_2 \begin{pmatrix} t \\ 1 \end{pmatrix}.
\]
(5) [10] Compute $e^{tA}$ for $A = \begin{pmatrix} 1 & -1 \\ -4 & 1 \end{pmatrix}$.

**Solution.** The characteristic polynomial of $A$ is

$$p(z) = z^2 - \text{tr}(A)z + \det(A) = z^2 - 2z - 3 = (z - 1)^2 - 4.$$  

The eigenvalues of $A$ are therefore $1 \pm 2$, whereby

$$e^{tA} = e^t \begin{bmatrix} I \cosh(2t) + (A - I) \frac{\sinh(2t)}{2} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cosh(2t) + \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix} \frac{\sinh(2t)}{2} \end{bmatrix}$$

$$= e^t \begin{pmatrix} \cosh(2t) & -\frac{1}{2} \sinh(2t) \\ -2 \sinh(2t) & \cosh(2t) \end{pmatrix}.$$  

**Alternative Solution.** The characteristic polynomial of $A$ is

$$p(z) = z^2 - \text{tr}(A)z + \det(A) = z^2 - 2z - 3 = (z + 1)(z - 3).$$  

The eigenvalues of $A$ are therefore $-1$ and $3$. Because

$$A + I = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}, \quad A - 3I = \begin{pmatrix} -2 & -1 \\ -4 & -2 \end{pmatrix},$$

Eigenpairs of $A$ are therefore

$$(-1, \begin{pmatrix} 1 \\ 2 \end{pmatrix}), \quad (3, \begin{pmatrix} -1 \\ 2 \end{pmatrix}).$$

Set $D = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}$ and $V = \begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix}$. Then $V = \frac{1}{4} \begin{pmatrix} 2 & 1 \\ -2 & 1 \end{pmatrix}$ and

$$e^{tA} = Ve^{tD}V^{-1} = \frac{1}{4} \begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{3t} \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -2 & 1 \end{pmatrix}$$

$$= \frac{1}{4} \begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 2e^{-t} & e^{-t} \\ -2e^{3t} & e^{3t} \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 2e^{-t} + 2e^{3t} & e^{-t} - e^{3t} \\ 4e^{-t} - 4e^{3t} & 2e^{-t} + 2e^{3t} \end{pmatrix}.$$  

(6) [5] Given that $e^{tA} = \begin{pmatrix} \cos(5t) + \frac{3}{5} \sin(5t) & \frac{4}{5} \sin(5t) \\ \frac{4}{5} \sin(5t) & \cos(5t) - \frac{3}{5} \sin(5t) \end{pmatrix}$, solve the initial-value problem

$$\frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}, \quad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$  

**Solution.** The solution is

$$x(t) = e^{tA} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} \cos(5t) + \frac{3}{5} \sin(5t) & \frac{4}{5} \sin(5t) \\ \frac{4}{5} \sin(5t) & \cos(5t) - \frac{3}{5} \sin(5t) \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos(5t) - \frac{1}{5} \sin(5t) \\ -\cos(5t) + \frac{7}{5} \sin(5t) \end{pmatrix}.$$
(7) Consider two interconnected tanks filled with brine (salt water). The first tank contains 60 liters and the second contains 40 liters. Brine flows with a concentration of 3 grams of salt per liter flows into the first tank at a rate of 5 liters per hour. Well stirred brine flows from the first tank to the second at a rate of 7 liters per hour, from the second to the first at a rate of 2 liters per hour, and from the second into a drain at a rate of 5 liters per hour. At \( t = 0 \) there are 40 grams of salt in the first tank and 20 grams in the second. Give an initial-value problem that governs the amount of salt in each tank as a function of time.

**Solution.** The rates work out so there will always be 60 liters of brine in the first tank and 40 liters in the second. Let \( S_1(t) \) and \( S_2(t) \) be the grams of salt in the first and second tanks respectively. These are governed by the initial-value problem

\[
\frac{dS_1}{dt} = 3 \cdot 5 + \frac{S_2}{40} - \frac{S_1}{60} \cdot 7, \quad S_1(0) = 40,
\]

\[
\frac{dS_2}{dt} = \frac{S_1}{60} \cdot 7 - \frac{S_2}{40} \cdot 2 - \frac{S_2}{40} \cdot 5, \quad S_2(0) = 20.
\]

(8) Consider the system

\[
\frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - y^2 \\ x - 6y + xy \end{pmatrix}.
\]

(a) Find all of its stationary points.

**Solution.** Stationary points satisfy

\[
0 = x - y^2, \quad 0 = x - 6y + xy.
\]

When the first equation is used to eliminate \( x \) in the second equation, one finds

\[
0 = y^3 + y^2 - 6y = y(y - 2)(y + 3),
\]

whereby either \( y = 0, \ y = 2, \) or \( y = -3 \). All the stationary points are therefore \( (0, 0), \ \ (4, 2), \ \ (9, -3) \).

(b) Compute the coefficient matrix of the linearization associated with each stationary point.

**Solution.** Because

\[
\begin{pmatrix} f(x, y) \\ g(x, y) \end{pmatrix} = \begin{pmatrix} x - y^2 \\ x - 6y + xy \end{pmatrix},
\]

the matrix of partial derivatives is

\[
\begin{pmatrix}
\frac{\partial x}{\partial x} f(x, y) & \frac{\partial y}{\partial x} f(x, y) \\
\frac{\partial x}{\partial y} g(x, y) & \frac{\partial y}{\partial y} g(x, y)
\end{pmatrix} = \begin{pmatrix} 1 & -2y \\ 1 + y & -6 + x \end{pmatrix}.
\]

Evaluating this matrix at each stationary point yields the coefficient matrices

\[
A = \begin{pmatrix} 1 & 0 \\ 1 & -6 \end{pmatrix} \quad \text{at} \quad (0, 0),
\]

\[
A = \begin{pmatrix} 1 & -4 \\ 3 & -2 \end{pmatrix} \quad \text{at} \quad (4, 2),
\]

\[
A = \begin{pmatrix} 1 & 6 \\ -2 & 3 \end{pmatrix} \quad \text{at} \quad (9, -3).
\]
(9) [16] Find a general solution for each of the following systems.

(a) \( \frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -5 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \)

**Solution.** The characteristic polynomial of \( A = \begin{pmatrix} 1 & 2 \\ -5 & -1 \end{pmatrix} \) is given by

\[ p(z) = z^2 - \text{tr}(A)z + \det(A) = z^2 + 9. \]

The eigenvalues of \( A \) are the roots of this polynomial, which are \( \pm 3 \). One therefore has

\[ e^{tA} = \begin{pmatrix} \cos(3t) + \frac{1}{3} \sin(3t) \\ \sin(3t) + \cos(3t) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos(3t) + \begin{pmatrix} 1 & 2 \\ -5 & -1 \end{pmatrix} \frac{3}{2} \sin(3t). \]

A general solution is therefore given by

\[ \begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} \cos(3t) + \frac{1}{3} \sin(3t) \\ \sin(3t) + \cos(3t) \end{pmatrix} + c_2 \begin{pmatrix} \frac{2}{3} \sin(3t) \\ \cos(3t) - \frac{1}{3} \sin(3t) \end{pmatrix}. \]

(b) \( \frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ -4 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \)

**Solution.** The characteristic polynomial of \( A = \begin{pmatrix} 0 & 2 \\ -4 & -4 \end{pmatrix} \) is given by

\[ p(z) = z^2 - \text{tr}(A)z + \det(A) = z^2 + 4z + 8 = (z + 2)^2 + 2^2. \]

The eigenvalues of \( A \) are the roots of this polynomial, which are \( -2 \pm i2 \). One therefore has

\[ e^{tA} = e^{-2t} \begin{pmatrix} \cos(2t) + (A + 2I) \frac{\sin(2t)}{2} \end{pmatrix} = e^{-2t} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos(2t) + \begin{pmatrix} 2 & 2 \\ -4 & -2 \end{pmatrix} \frac{2}{2} \sin(2t). \]

A general solution is therefore given by

\[ \begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{-2t} \begin{pmatrix} \cos(2t) + \sin(2t) \\ -2 \sin(2t) \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} \sin(2t) \\ \cos(2t) - \sin(2t) \end{pmatrix}. \]
(10) [8] Sketch the phase portrait for each of the systems in the previous problem. Identify
the type and stability of the origin.

**Solution.** The characteristic polynomial of $A$ is $p(z) = z^2 + 9$. Because $\mu = 0,$
$\delta = -9 < 0,$ and $a_{21} < 0$ the phase portrait is a *clockwise center*. The origin is
thereby *stable*. The phase portrait should indicate a family of clockwise elliptical
trajectories that go around the origin.

**Solution.** The characteristic polynomial of $A$ is $p(z) = (z + 2)^2 + 4$. Because
$\mu = -2, \delta = -4 < 0,$ and $a_{21} < 0$ the phase portrait is a *clockwise spiral sink*. The
origin is therefore *asymptotically stable*. The phase portrait should indicate a family
of clockwise spiral trajectories that approach the origin.

(11) [9] Suppose you know that for some first-order planar system of nonlinear ordinary
differential equations:
- its stationary points are $(0, 0), (2, 2),$ and $(4, 0);$ 
- for $(0, 0)$ the coefficient matrix of the linearization has eigenvalues $1$ and $2$ with
respective eigenvectors
  \[
  \begin{pmatrix}
  1 \\
  0
  \end{pmatrix}
  \quad \text{and} \quad \begin{pmatrix}
  1 \\
  1
  \end{pmatrix};
  \]
- for $(2, 2)$ the coefficient matrix of the linearization has eigenvalues $-1$ and $-2$
with respective eigenvectors
  \[
  \begin{pmatrix}
  1 \\
  -1
  \end{pmatrix}
  \quad \text{and} \quad \begin{pmatrix}
  1 \\
  1
  \end{pmatrix};
  \]
- for $(4, 0)$ the coefficient matrix of the linearization has eigenvalues $1$ and $-1$ with
respective eigenvectors
  \[
  \begin{pmatrix}
  1 \\
  -1
  \end{pmatrix}
  \quad \text{and} \quad \begin{pmatrix}
  1 \\
  0
  \end{pmatrix}.
  \]

Sketch a plausible phase portrait for the system. Identify the type and stability of
each stationary point.

**Solution.** For each stationary point you have the following.
- The stationary point $(0, 0)$ has two positive simple real eigenvalues. It therefore
is a *nodal source* and thereby is *unstable*. Near it there is one trajectory that
emerges from $(0, 0)$ tangent to each side of the line $y = x$. Every other trajectory
emerges from $(0, 0)$ tangent to the line $y = 0$.
- The stationary point $(2, 2)$ has two negative simple real eigenvalues. It therefore
is a *nodal sink* and thereby is *asymptotically stable*. Near it there is one trajectory
that approaches $(2, 2)$ tangent to each side of the line $y = x$. Every other
trajectory approaches $(2, 2)$ tangent to the line $y = -x + 4$.
- The stationary point $(4, 0)$ has one negative and one positive real eigenvalue. It
therefore is a *saddle* and thereby is *unstable*. Near it there is one trajectory that
emerges from $(4, 0)$ tangent to each side of the line $y = -x + 4$. There is also
one trajectory that approaches $(4, 0)$ tangent to each side of the line $y = 0$. 