(1) Consider the differential equation \( \frac{dy}{dt} = y^2(4 - y^2) \) over the interval \(-5 \leq y \leq 5\).

(a) Sketch its phase-line portrait in this interval.
(b) Identify its equilibrium (stationary) points and discuss their stability.
(c) If \( y(0) = 3 \), how does the solution \( y(t) \) behave as \( t \to \infty \)?

**Solution (a,b):** The right-hand side factors as \( y^2(2 + y)(2 - y) \). The stationary solutions are \( y = -2, y = 0, \) and \( y = 2 \). A sign analysis of \( y^2(2 + y)(2 - y) \) shows that the phase-line portrait for this equation is therefore

```
-  +  +  -
\-
-2  0  2
```
unstable semistable stable

**Solution (c):** The phase-line shows that if \( y(0) = 3 \) then \( y(t) \to 2 \) as \( t \to \infty \).

(2) Solve (possibly implicitly) each of the following initial-value problems.

(a) \( \frac{dy}{dt} + \frac{2ty}{1 + t^2} = t^2, \quad y(0) = 1 \).

**Solution:** This equation is linear and is already in normal form. An integrating factor is
\[
\exp\left( \int_0^t \frac{2s}{1 + s^2} \, ds \right) = \exp( \log(1 + t^2) ) = 1 + t^2,
\]
so that
\[
\frac{d}{dt}((1 + t^2)y) = (1 + t^2)t^2 = t^2 + t^4.
\]
Integrate this to obtain
\[
(1 + t^2)y = \frac{1}{2}t^3 + \frac{1}{4}t^5 + C.
\]
The initial condition \( y(0) = 1 \) implies that \( C = (1 + 0^2) \cdot 1 - \frac{1}{2}0^3 - \frac{1}{4}0^5 = 1 \). Therefore
\[
y = \frac{1 + \frac{1}{2}t^3 + \frac{1}{4}t^5}{1 + t^2}.
\]

(b) \( \frac{dy}{dx} + \frac{e^x y + 2x}{2y + e^x} = 0, \quad y(0) = 0 \).

**Solution:** Express this equation in the differential form
\[
(e^x y + 2x) \, dx + (2y + e^x) \, dy = 0.
\]
This differential form is exact because
\[
\partial_y(e^x y + 2x) = e^x \quad \text{and} \quad \partial_x(2y + e^x) = e^x.
\]
We can therefore find \( H(x, y) \) such that
\[
\partial_x H(x, y) = e^x y + 2x, \quad \partial_y H(x, y) = 2y + e^x.
\]
The first equation implies \( H(x, y) = e^x y + x^2 + h(y) \). Plugging this into the second equation gives
\[
e^x + h'(y) = 2y + e^x,
\]
which yields \( h'(y) = 2y \). Taking \( h(y) = y^2 \), the general solution is
\[
e^x y + x^2 + y^2 = C.
\]
The initial condition \( y(0) = 0 \) implies that \( C = e^0 \cdot 0 + 0^2 + 0^2 = 0 \). Therefore
\[
y^2 + e^x y + x^2 = 0.
\]
If you had been asked for an explicit solution then the quadratic formula yields
\[
y = -e^x + \sqrt{e^{2x} - 4x^2}.
\]
Here the positive square root is taken because that solution satisfies the initial condition. It exists wherever \( e^{2x} \geq 4x^2 \).

(3) Let \( y(t) \) be the solution of the initial-value problem
\[
\frac{dy}{dt} = y^2 + t^2, \quad y(0) = 1.
\]
Use one step of the improved Euler (trapezoidal-Heun) method to approximate \( y(0.1) \).

**Solution.** The improved Euler method is
\[
f_n = f(y_n, t_n),
\]
\[
\tilde{y}_{n+1} = y_n + hf_n,
\]
\[
t_{n+1} = t_n + h,
\]
\[
\tilde{f}_{n+1} = f(\tilde{y}_{n+1}, t_{n+1}),
\]
\[
y_{n+1} = y_n + \frac{h}{2}(f_n + \tilde{f}_{n+1}),
\]
where \( h \) is the time step, \( t_0 \) is the initial time, and \( y_0 \) is the initial data.

When the improved Euler method is applied with \( h = 0.1, t_0 = 0, y_0 = 1 \), and \( f(y, t) = y^2 + t^2 \) for one step
\[
f_0 = f(y_0, t_0) = y_0^2 + t_0^2 = 1^2 + 0^2 = 1,
\]
\[
\tilde{y}_1 = y_0 + hf_n = 1 + 0.1 \cdot 1 = 1.1,
\]
\[
t_1 = t_0 + h = 0 + 0.1 = 0.1,
\]
\[
\tilde{f}_1 = f(\tilde{y}_1, t_1) = \tilde{y}_1^2 + t_1^2 = (1.1)^2 + (0.1)^2,
\]
\[
y_1 = y_0 + \frac{h}{2}(f_0 + \tilde{f}_1) = 1 + 0.05(1 + (1.1)^2 + (0.1)^2).
\]
The approximation is therefore
\[
y(0.1) \approx y_1 = 1 + 0.05(1 + (1.1)^2 + (0.1)^2).
\]
You DO NOT have to work out the arithmetic! If you did then \( y_1 = 1.111 \).
(4) Give an explicit general solution of the following equations.

(a) \( \frac{d^2 y}{dt^2} - 2\frac{dy}{dt} + 5y = te^t + \cos(2t) \)

**Solution.** This is a constant coefficient, inhomogeneous, linear equation. Its characteristic polynomial is

\[
P(z) = z^2 - 2z + 5 = (z - 1)^2 + 4 = (z - 1)^2 + 2^2.
\]

This has the conjugate pair of roots \(1 \pm i2\), which yields a general solution of the associated homogeneous problem

\[
y_H(t) = c_1 e^t \cos(2t) + c_2 e^t \sin(2t).
\]

A particular solution \(y_P(t)\) can be found by the method of undetermined coefficients. The characteristics of the forcing terms \(te^t\) and \(\cos(2t)\) are \(r + is = 1\) and \(r + is = i2\) respectively. Because these characteristics are different, they should be treated separately. This can be done using either KEY identity evaluation (as in the lectures) or direct substitution (as in the book).

**KEY Identity Evaluations.** The forcing term \(te^t\) has degree \(d = 1\) and characteristic \(r + is = 1\), which is a root of \(P(z)\) of multiplicity \(m = 0\). Because \(m + d = 1\), you need the KEY identity and its first derivative

\[
L(e^{zt}) = (z^2 - 2z + 5)e^{zt},
\]

\[
L(te^{zt}) = (z^2 - 2z + 5)t e^{zt} + (2z - 2) e^{zt}.
\]

Evaluate these at \(z = 1\) to find \(L(e^t) = 4e^t\) and \(L(te^t) = 4t e^t\). Dividing the second of these equations by 4 yields \(L(\frac{1}{4}te^t) = t e^t\), which implies \(y_{P1}(t) = \frac{1}{4}t e^t\).

The forcing term \(\cos(2t)\) has degree \(d = 0\) and characteristic \(r + is = i2\), which is a root of \(P(z)\) of multiplicity \(m = 0\). Because \(m + d = 0\), you only need the KEY identity

\[
L(e^{zt}) = (z^2 - 2z + 5)e^{zt}.
\]

Evaluate this at \(z = i2\) to find \(L(e^{izt}) = (1 + i4)e^{izt}\). Dividing this by \((1 - i4)\) yields

\[
L\left(\frac{e^{2t}}{1 - i4}\right) = e^{izt}.
\]

Because \(\cos(2t) = \text{Re}(e^{izt})\), the above equation implies

\[
y_{P2}(t) = \text{Re}\left(\frac{e^{2t}}{1 - i4}\right) = \text{Re}\left(\frac{(1 + i4)e^{izt}}{1^2 + 4^2}\right) = \frac{1}{17}\text{Re}\left((1 + i4)e^{izt}\right) = \frac{1}{17}\left(\cos(2t) - 4\sin(2t)\right).
\]

Combining these particular solutions with the general solution of the associated homogeneous problem yields the general solution

\[
y = y_H(t) + y_{P1}(t) + y_{P2}(t)
\]

\[
= c_1 e^t \cos(2t) + c_2 e^t \sin(2t) + \frac{1}{4}t e^t + \frac{1}{17} \cos(2t) - \frac{4}{17} \sin(2t).
\]
Direct Substitution. The forcing term $te^t$ has degree $d = 1$ and characteristic $r + is = 1$, which is a root of $P(z)$ of multiplicity $m = 0$. Because $m = 0$ and $m + d = 1$, you seek a particular solution of the form
\[ y_{P1}(t) = A_0te^t + A_1e^t. \]
Because
\[ y'_{P1}(t) = A_0te^t + (A_0 + A_1)e^t, \quad y''_{P1}(t) = A_0te^t + (2A_0 + A_1)e^t, \]
one sees that
\[ Ly_{P1}(t) = y''_{P1}(t) - 2y'_{P1}(t) + 5y_{P1}(t) \]
\[ = (A_0t e^t + (2A_0 + A_1)e^t) - 2(A_0te^t + (A_0 + A_1)e^t) \]
\[ + 5(A_0te^t + A_1e^t) \]
\[ = 4A_0te^t + 4A_1e^t. \]
Setting $4A_0te^t + 4A_1e^t = t e^t$, we see that $4A_0 = 1$ and $4A_1 = 0$, whereby $A_0 = \frac{1}{4}$ and $A_1 = 0$. Hence, $y_{P}(t) = \frac{1}{4}t e^t$.
The forcing term $\cos(2t)$ has degree $d = 0$ and characteristic $r + is = i2$, which is a root of $P(z)$ of multiplicity $m = 0$. Because $m = 0$ and $m + d = 0$, you seek a particular solution of the form
\[ y_{P2}(t) = A \cos(2t) + B \sin(2t). \]
Because
\[ y'_{P2}(t) = -2A \sin(2t) + 2B \cos(2t), \quad y''_{P2}(t) = -4A \cos(2t) - 4B \sin(2t), \]
one sees that
\[ Ly_{P2}(t) = y''_{P2}(t) - 2y'_{P2}(t) + 5y_{P2}(t) \]
\[ = ( - 4A \cos(2t) - 4B \sin(2t)) - 2( - 2A \sin(2t) + 2B \cos(2t)) \]
\[ + 5(A \cos(2t) + B \sin(2t)) \]
\[ = (A - 4B) \cos(2t) + (B + 4A) \sin(2t). \]
Setting $(A - 4B) \cos(2t) + (B + 4A) \sin(2t) = \cos(2t)$, we see that
\[ A - 4B = 1, \quad B + 4A = 0. \]
This system can be solved by any method you choose to find $A = \frac{1}{17}$ and $B = -\frac{4}{17}$, whereby
\[ y_{P2}(t) = \frac{1}{17} \cos(2t) - \frac{4}{17} \sin(2t). \]
Combining these particular solutions with the general solution of the associated homogeneous problem yields the general solution
\[ y = y_{H}(t) + y_{P1}(t) + y_{P2}(t) \]
\[ = c_1e^t \cos(2t) + c_2e^t \sin(2t) + \frac{1}{4}t e^t + \frac{1}{17} \cos(2t) - \frac{4}{17} \sin(2t). \]
(b) \( \frac{d^2y}{dt^2} + y = \tan(t) \)

**Solution.** This is a constant coefficient, inhomogeneous, linear equation. Its characteristic polynomial is

\[ P(z) = z^2 + 1 = z^2 + 1. \]

This has the conjugate pair of roots \( \pm i \), which yields a general solution of the associated homogeneous problem

\[ y_H(t) = c_1 \cos(t) + c_2 \sin(t). \]

Because of the form of the forcing term, you must use the method of variation of parameters to find a particular solution. The equation is already in normal form. Seek a solution in the form

\[ y_H(t) = u_1(t) \cos(t) + u_2(t) \sin(t), \]

where \( u_1'(t) \) and \( u_2'(t) \) satisfy

\begin{align*}
  u_1'(t) \cos(t) + u_2'(t) \sin(t) &= 0, \\
  -u_1'(t) \sin(t) + u_2'(t) \cos(t) &= \tan(t).
\end{align*}

Solve this system to find

\[ u_1'(t) = -\frac{\sin(t)^2}{\cos(t)} = \cos(t) - \sec(t), \quad u_2'(t) = \sin(t). \]

Integrate these to find

\[ u_1(t) = c_1 + \sin(t) - \log(\tan(t) + \sec(t)), \quad u_2(t) = c_2 - \cos(t). \]

A general solution is therefore

\[ y = c_1 \cos(t) + c_2 \sin(t) - \cos(t) \log(\tan(t) + \sec(t)). \]

(5) When a mass of 3 kilograms is hung vertically from a spring, it stretches the spring 0.25 meters. (Gravitational acceleration is 9.8 m/sec\(^2\).) At \( t = 0 \) the mass is set in motion from 0.5 meters below its equilibrium (rest) position with an upward velocity of 2 m/sec. Neglect drag and assume that the spring force is proportional to its displacement. Formulate an initial-value problem that governs the motion of the mass for \( t > 0 \). (DO NOT solve this initial-value problem; just write it down!)

**Solution.** Let \( h(t) \) be the displacement (in meters) of the mass from its equilibrium (rest) position at time \( t \) (in seconds), with upward displacements being positive. The governing initial-value problem then has the form

\[ m \frac{d^2h}{dt^2} + kh = 0, \quad h(0) = -0.5, \quad h'(0) = 2, \]

where \( m \) is the mass and \( k \) is the spring constant. The problem says that \( m = 3 \) kilograms. The spring constant is obtained by balancing the weight of the mass (\( mg = 3 \cdot 9.8 \) Newtons) with the force applied by the spring when it is stretched .25 m. This gives \( k \cdot .25 = 3 \cdot 9.8 \), or

\[ k = \frac{3 \cdot 9.8}{.25} = 12 \cdot 9.8 \quad \text{Newtons/m}. \]
The governing initial-value problem is therefore

\[ 3 \frac{d^2 h}{dt^2} + 12 \cdot 9.8 h = 0, \quad h(0) = -0.5, \quad h'(0) = 2. \]

Had you chosen positive \( h \) to be downward displacements then the only thing that would differ is the sign of the initial data.

(6) Give an explicit general solution of the equation

\[ \frac{d^2 y}{dt^2} + 2 \frac{dy}{dt} + 10y = 0. \]

Sketch a typical solution. If this equation governs a damped spring-mass system, is the system over, under, or critically damped?

**Solution.** This is a constant coefficient, inhomogeneous, linear equation. Its characteristic polynomial is

\[ P(z) = z^2 + 2z + 10 = (z + 1)^2 + 3^2. \]

This has the conjugate pair of roots \(-1 \pm i3\), which yields a general solution

\[ y = c_1 e^{-t} \cos(3t) + c_2 e^{-t} \sin(3t). \]

When \( c_1^2 + c_2^2 > 0 \) this can be put into the amplitude-phase form

\[ y = Ae^{-t} \cos(3t - \delta), \]

where \( A > 0 \) and \( 0 \leq \delta < 2\pi \) are determined from \( c_1 \) and \( c_2 \) by

\[ A = \sqrt{c_1^2 + c_2^2}, \quad \cos(\delta) = \frac{c_1}{A}, \quad \sin(\delta) = \frac{c_2}{A}. \]

The sketch should show a decaying oscillation with amplitude \( Ae^{-t} \) and quasiperiod \( \frac{2\pi}{3} \). This corresponds to an under damped spring-mass system.

(7) Find the Laplace transform \( Y(s) \) of the solution \( y(t) \) to the initial-value problem

\[ \frac{d^2 y}{dt^2} + 4 \frac{dy}{dt} + 8y = g(t), \quad y(0) = 2, \quad y'(0) = 4. \]

where

\[ g(t) = \begin{cases} 4 & \text{for } 0 \leq t < 2, \\ t^2 & \text{for } 2 \leq t. \end{cases} \]

You may refer to the table in Section 6.2 of the book. (DO NOT take the inverse Laplace transform to find \( y(t) \); just solve for \( Y(s) \)!)  

**Solution.** The Laplace transform of the initial-value problem is

\[ \mathcal{L}[y''](s) + 4\mathcal{L}[y'](s) + 8\mathcal{L}[y](s) = \mathcal{L}[g](s), \]

where

\[ \mathcal{L}[y](s) = Y(s), \]
\[ \mathcal{L}[y'](s) = sY(s) - y(0) = sY(s) - 2, \]
\[ \mathcal{L}[y''](s) = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - 2s - 4. \]
To compute $\mathcal{L}[g](s)$, first write $g$ as
\[
g(t) = (1 - u(t - 2))4 + u(t - 2)t^2 = 4 - u(t - 2)4 + u(t - 2)t^2
\]
\[
= 4 + u(t - 2)(t^2 - 4) = 4 + u(t - 2)((2 + (t - 2))^2 - 4)
\]
\[
= 4 + u(t - 2)(4(t - 2) + (t - 2)^2).
\]
Referring to the table of Laplace transforms in the book, item 13 with $c = 2$, item 1, and item 3 with $n = 1$ and $n = 2$ then show that
\[
\mathcal{L}[g](s) = 4\mathcal{L}[1](s) + 4\mathcal{L}[u(t - 2)(t - 2)](s) + \mathcal{L}[u(t - 2)(t - 2)^2](s)
\]
\[
= 4\mathcal{L}[1](s) + 4e^{-2s}\mathcal{L}[t](s) + e^{-2s}\mathcal{L}[t^2](s)
\]
\[
= \frac{4}{s} + 4e^{-2s}\frac{1}{s^2} + e^{-2s}\frac{2}{s^3}.
\]
The Laplace transform of the initial-value problem then becomes
\[
(s^2Y(s) - 2s - 4) + 4(sY(s) - 2) + 8Y(s) = \frac{4}{s} + e^{-2s}\frac{4}{s^2} + e^{-2s}\frac{2}{s^3},
\]
which becomes
\[
(s^2 + 4s + 8)Y(s) - 2s - 12 = \frac{4}{s} + e^{-2s}\frac{4}{s^2} + e^{-2s}\frac{2}{s^3}.
\]
Hence, $Y(s)$ is given by
\[
Y(s) = \frac{1}{s^2 + 4s + 8}
\]
\[
\left(2s + 12 + \frac{4}{s} + e^{-2s}\frac{4}{s^2} + e^{-2s}\frac{2}{s^3}\right).
\]

(8) Find the function $y(t)$ whose Laplace transform $Y(s)$ is given by $Y(s) = \frac{e^{-3s}4}{s^2 - 6s + 5}$.
You may refer to the table in Section 6.2 of the book.

**Solution.** The denominator factors as $(s - 5)(s - 1)$, so the partial fraction decomposition is
\[
\frac{4}{s^2 - 6s + 5} = \frac{4}{(s - 5)(s - 1)} = \frac{1}{s - 5} - \frac{1}{s - 1}.
\]
Referring to the table of Laplace transforms in the book, item 11 with $n = 0$ and $a = 5$, and with $n = 0$ and $a = 1$ gives
\[
\mathcal{L}[e^{5t}](s) = \frac{1}{s - 5}, \quad \mathcal{L}[e^t](s) = \frac{1}{s - 1},
\]
whereby
\[
\frac{4}{s^2 - 6s + 5} = \mathcal{L}[e^{5t}](s) - \mathcal{L}[e^t](s) = \mathcal{L}[e^{5t} - e^t](s).
\]
It follows from item 13 with $c = 3$ that
\[
\mathcal{L}[u(t - 3)(e^{5(t-3)} - e^{t-3})](s) = e^{-3s}\frac{4}{s^2 - 6s + 5} = Y(s).
\]
You therefore conclude that
\[
y(t) = \mathcal{L}^{-1}[Y(s)](t) = u(t - 3)(e^{5(t-3)} - e^{t-3}).
\]
(9) Give a general solution of \[ \frac{dx}{dt} = Ax \] for the following \( A \).

(a) \( A = \begin{pmatrix} 6 & 4 \\ 4 & 0 \end{pmatrix} \)

**Solution.** The characteristic polynomial of \( A \) is

\[
p(z) = z^2 - \text{tr}(A)z + \det(A) \\
= z^2 - 6z - 16 = (z - 3)^2 - 25 = (z - 3)^2 - 5^2.
\]

The eigenvalues of \( A \) are the roots of this polynomial, which are \( 3 \pm 5 \), or simply \(-2 \) and \( 8 \). One therefore has

\[
e^{tA} = e^{3t} \begin{bmatrix} 1 & 3 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} \cosh(5t) & \frac{4}{5}\sinh(5t) \\ \frac{4}{5}\sinh(5t) & \cosh(5t) - \frac{3}{5}\sinh(5t) \end{bmatrix}.
\]

A general solution is therefore given by

\[
x = c_1 e^{3t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 4 \\ 2 \end{bmatrix}.
\]

**Alternative Solution.** The characteristic polynomial of \( A \) is

\[
p(z) = z^2 - \text{tr}(A)z + \det(A) \\
= z^2 - 6z - 16 = (z - 3)^2 - 25 = (z - 3)^2 - 5^2.
\]

The eigenvalues of \( A \) are the roots of this polynomial, which are \( 3 \pm 5 \), or simply \(-2 \) and \( 8 \). Because

\[
A + 2I = \begin{pmatrix} 8 & 4 \\ 4 & 2 \end{pmatrix}, \quad A - 8I = \begin{pmatrix} -2 & 4 \\ 4 & -8 \end{pmatrix},
\]

we see that \( A \) has the eigenpairs

\[
\begin{pmatrix} -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 8 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}.
\]

A general solution is therefore given by

\[
x = c_1 e^{-2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}.
\]
(b) \( A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \)

**Solution.** The characteristic polynomial of \( A \) is

\[
p(z) = z^2 - \text{tr}(A)z + \det(A) \\
= z^2 - 2z + 5 = (z - 1)^2 + 4 = (z - 1)^2 + 2^2.
\]

The eigenvalues of \( A \) are the roots of this polynomial, which are \( 1 \pm i2 \). One therefore has

\[
e^{tA} = e^t \left[ \begin{pmatrix} 1 \cos(2t) + (A - I) \frac{\sin(2t)}{2} \end{pmatrix} \right]
\]

\[
e^t \left[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos(2t) + \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} \frac{\sinh(2t)}{2} \right]
\]

\[
e^t \left( \begin{pmatrix} \cos(2t) & \sin(2t) \\ -\sin(2t) & \cos(2t) \end{pmatrix} \right)
\]

A general solution is therefore given by

\[
x = c_1 e^t \begin{pmatrix} \cos(2t) \\ -\sin(2t) \end{pmatrix} + c_2 e^t \begin{pmatrix} \sin(2t) \\ \cos(2t) \end{pmatrix}.
\]

**Alternative Solution.** The characteristic polynomial of \( A \) is

\[
p(z) = z^2 - \text{tr}(A)z + \det(A) \\
= z^2 - 2z + 5 = (z - 1)^2 + 4 = (z - 1)^2 + 2^2.
\]

The eigenvalues of \( A \) are the roots of this polynomial, which are \( 1 \pm i2 \). Because

\[
A - (1 + i2)I = \begin{pmatrix} -i2 & 2 \\ -2 & -i2 \end{pmatrix}, \quad A - (1 - i2)I = \begin{pmatrix} i2 & 2 \\ -2 & i2 \end{pmatrix},
\]

we see that \( A \) has the eigenpairs

\[
\left( 1 + i2, \begin{pmatrix} 1 \\ i \end{pmatrix} \right), \quad \left( 1 - i2, \begin{pmatrix} -i \\ 1 \end{pmatrix} \right).
\]

Because

\[
e^{(1+i2)t} \begin{pmatrix} 1 \\ i \end{pmatrix} = e^t \begin{pmatrix} \cos(2t) + i \sin(2t) \\ -\sin(2t) + i \cos(2t) \end{pmatrix},
\]

two real solutions of the system are

\[
e^t \begin{pmatrix} \cos(2t) \\ -\sin(2t) \end{pmatrix}, \quad e^t \begin{pmatrix} \sin(2t) \\ \cos(2t) \end{pmatrix}.
\]

A general solution is therefore

\[
x = c_1 e^t \begin{pmatrix} \cos(2t) \\ -\sin(2t) \end{pmatrix} + c_2 e^t \begin{pmatrix} \sin(2t) \\ \cos(2t) \end{pmatrix}.
\]
(10) A matrix $A$ has eigenvalues $-2$ and $-1$ with associated eigenvectors $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$.

(a) Give a general solution to $\frac{dx}{dt} = Ax$.

**Solution.** A general solution is
\[ x = c_1 e^{-2t} \begin{pmatrix} 3 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} -1 \\ 2 \end{pmatrix}. \]

(b) Sketch a phase-plane portrait for this system. Identify its type. (Carefully mark the exact lines, curves, and arrows!)

**Solution.** The coefficient matrix has two negative eigenvalues. The origin is therefore a *nodal sink* and is thereby *asymptotically stable*. There is one trajectory that approaches $(0,0)$ along each half of the lines $x = 3y$ and $y = -2x$. (These are the lines of eigenvectors.) Every other trajectory approaches $(0,0)$ tangent to the line $y = -2x$, which is the line corresponding to the eigenvalue with the smaller absolute value.

(11) Consider the nonlinear system
\[ \frac{dx}{dt} = -5y, \]
\[ \frac{dy}{dt} = x - 4y - x^2. \]

(a) Find all of its equilibrium (critical, stationary) points.

**Solution.** Stationary points satisfy
\[ 0 = -5y, \quad 0 = x - 4y - x^2. \]

The first equation implies $y = 0$, whereby the second equation becomes $0 = x - x^2 = x(1-x)$, which implies either $x = 0$ or $x = 1$. All the stationary points of the system are therefore
\[ (0,0), \quad (1,0). \]

(b) Compute the coefficient matrix of the linearization (the derivative matrix) at each equilibrium (critical, stationary) point.

**Solution.** Because
\[ \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix} = \begin{pmatrix} -5y \\ x - 4y - x^2 \end{pmatrix}, \]
the matrix of partial derivatives is
\[ \begin{pmatrix} \frac{\partial f}{\partial x}(x,y) & \frac{\partial f}{\partial y}(x,y) \\ \frac{\partial g}{\partial x}(x,y) & \frac{\partial g}{\partial y}(x,y) \end{pmatrix} = \begin{pmatrix} 0 & -5 \\ 1 - 2x & -4 \end{pmatrix}. \]

Evaluating this matrix at each stationary point yields the coefficient matrices
\[ A = \begin{pmatrix} 0 & -5 \\ 1 & -4 \end{pmatrix} \text{ at } (0,0), \quad A = \begin{pmatrix} 0 & -5 \\ -1 & -4 \end{pmatrix} \text{ at } (1,0). \]
(c) Identify the type and stability of each equilibrium (critical, stationary) point.

**Solution.** The coefficient matrix $A$ at $(0,0)$ has eigenvalues that satisfy

$$0 = \det(zI - A) = z^2 - \text{tr}(A)z + \det(A) = z^2 + 4z + 5 = (z + 2)^2 + 1^2.$$

The eigenvalues are thereby $-2 \pm i$. Because $a_{21} = 1 > 0$, the stationary point $(0,0)$ is therefore a *counter clockwise spiral sink*, which is *asymptotically stable*. This is one of the generic types, so it describes the phase-plane portrait of the nonlinear system near $(0,0)$.

The coefficient matrix $A$ at $(1,0)$ has eigenvalues that satisfy

$$0 = \det(zI - A) = z^2 - \text{tr}(A)z + \det(A) = z^2 + 4z - 5 = (z + 2)^2 - 3^2.$$

The eigenvalues are thereby $-2 \pm 3$, or simply $-5$ and $1$. The stationary point $(1,0)$ is therefore a *saddle*, which is *unstable*. This is one of the generic types, so it describes the phase-plane portrait of the nonlinear system near $(1,0)$.

(d) Sketch a plausible global phase-plane portrait. (Carefully mark the exact lines, curves, and arrows!)

**Solution.** The stationary point $(0,0)$ is a *counter clockwise spiral sink*.

The stationary point $(1,0)$ is a *saddle*. The coefficient matrix $A$ has eigenvalues $-5$ and $1$. Because

$$A + 5I = \begin{pmatrix} 5 & -5 \\ -1 & 1 \end{pmatrix}, \quad A - I = \begin{pmatrix} -1 & -5 \\ -1 & -5 \end{pmatrix},$$

it has the eigenpairs

$$\left( -5, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right), \quad \left( 1, \begin{pmatrix} -5 \\ 1 \end{pmatrix} \right).$$

Near $(1,0)$ there is one trajectory that emerges from $(1,0)$ tangent to each side of the line $x = 1 - 5y$. There is also one trajectory that approaches $(1,0)$ tangent to each side of the line $y = x - 1$.

(12) Consider the nonlinear system

$$\frac{dx}{dt} = x(3 - 3x + 2y), \quad \frac{dy}{dt} = y(6 - x - y).$$

(a) Find all of its equilibrium (critical, stationary) points.

**Solution.** Stationary points satisfy

$$0 = x(3 - 3x + 2y), \quad 0 = y(6 - x - y).$$

The first equation implies either $x = 0$ or $3 - 3x + 2y = 0$, while the second equation implies either $y = 0$ or $6 - x - y = 0$. If $x = 0$ and $y = 0$ then $(0,0)$ is a stationary point. If $x = 0$ and $6 - x - y = 0$ then $(0,6)$ is a stationary point. If $3 - 3x + 2y = 0$ and $y = 0$ then $(1,0)$ is a stationary point. If $3 - 3x + 2y = 0$ and $6 - x - y = 0$ then upon solving these equations one finds that $(3,3)$ is a stationary point. All the stationary points of the system are therefore

$$(0,0), \quad (0,6), \quad (1,0), \quad (3,3).$$
(b) Compute the coefficient matrix of the linearization (the derivative matrix) at each equilibrium (critical, stationary) point.

**Solution.** Because

\[
\begin{pmatrix}
  f(x, y) \\
  g(x, y)
\end{pmatrix}
= 
\begin{pmatrix}
  3x - 3x^2 + 2xy \\
  6y - xy - y^2
\end{pmatrix},
\]

the matrix of partial derivatives is

\[
\begin{pmatrix}
  \frac{\partial f}{\partial x}(x, y) & \frac{\partial f}{\partial y}(x, y) \\
  \frac{\partial g}{\partial x}(x, y) & \frac{\partial g}{\partial y}(x, y)
\end{pmatrix}
= 
\begin{pmatrix}
  3 - 6x + 2y & 2x \\
  -y & 6 - x - 2y
\end{pmatrix}.
\]

Evaluating this matrix at each stationary point yields the coefficient matrices

\[
A = \begin{pmatrix}
  3 & 0 \\
  0 & 6
\end{pmatrix} \quad \text{at} \quad (0, 0), \quad A = \begin{pmatrix}
  15 & 0 \\
  -6 & -6
\end{pmatrix} \quad \text{at} \quad (0, 6),
\]

\[
A = \begin{pmatrix}
  -3 & 2 \\
  0 & 5
\end{pmatrix} \quad \text{at} \quad (1, 0), \quad A = \begin{pmatrix}
  -9 & 6 \\
  -3 & -3
\end{pmatrix} \quad \text{at} \quad (3, 3). 
\]

(c) Identify the type and stability of each equilibrium (critical, stationary) point.

**Solution.** The coefficient matrix \( A \) at \((0, 0)\) is diagonal, so you can read-off its eigenvalues as 3 and 6. The stationary point \((0, 0)\) is thereby a *nodal source*, which is *unstable*. This is one of the generic types, so it describes the phase-plane portrait of the nonlinear system near \((0, 0)\).

The coefficient matrix \( A \) at \((0, 6)\) is triangular, so you can read-off its eigenvalues as \(-6\) and 15. The stationary point \((0, 6)\) is thereby a *saddle*, which is *unstable*. This is one of the generic types, so it describes the phase-plane portrait of the nonlinear system near \((0, 6)\).

The coefficient matrix \( A \) at \((1, 0)\) is triangular, so you can read-off its eigenvalues as \(-3\) and 5. The stationary point \((1, 0)\) is thereby a *saddle*, which is *unstable*. This is one of the generic types, so it describes the phase-plane portrait of the nonlinear system near \((1, 0)\).

The coefficient matrix \( A \) at \((0, 6)\) has eigenvalues that satisfy

\[
0 = \det(zI - A) = z^2 - \text{tr}(A)z + \det(A) = z^2 + 12z + 45 = (z + 6)^2 + 3^2.
\]

Its eigenvalues are thereby \(-6 \pm i3\). Because \(a_{21} = -3 < 0\), the stationary point \((3, 3)\) is therefore a *clockwise spiral sink*, which is *asymptotically stable*. This is one of the generic types, so it describes the phase-plane portrait of the nonlinear system near \((3, 3)\).

(d) Sketch a plausible global phase-plane portrait. (Carefully mark the exact lines, curves, and arrows!)

**Solution.** First observe that the lines \(x = 0\) and \(y = 0\) are invariant. A trajectory that starts on one of these lines must stay on that line. Along the line \(x = 0\) the system reduces to

\[
\frac{dy}{dt} = y(6 - y).
\]
Along the line \( y = 0 \) the system reduces to
\[
\frac{dx}{dt} = 3x(1 - x) .
\]
The arrows along these invariant lines can be determined from a phase-line portrait of these reduced systems.

The stationary point \((0, 0)\) is a *nodal source*. The coefficient matrix \(A\) has eigenvalues 3 and 6. Because
\[
A - 3I = \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}, \quad A - 6I = \begin{pmatrix} -3 & 0 \\ 0 & 0 \end{pmatrix},
\]

it has the eigenpairs
\[
(3, \begin{pmatrix} 1 \\ 0 \end{pmatrix}), \quad (6, \begin{pmatrix} 0 \\ 1 \end{pmatrix}).
\]

Near there is one trajectory that emerges from \((0, 0)\) along each side of the invariant lines \( y = 0 \) and \( x = 0 \). Every other trajectory emerges from \((0, 0)\) tangent to the line \( y = 0 \), which is the line corresponding to the eigenvalue with the smaller absolute value.

The stationary point \((0, 6)\) is a *saddle*. The coefficient matrix \(A\) has eigenvalues \(-6\) and 15. Because
\[
A + 6I = \begin{pmatrix} 21 & 0 \\ -6 & 0 \end{pmatrix}, \quad A - 15I = \begin{pmatrix} 0 & 0 \\ -6 & -21 \end{pmatrix},
\]

it has the eigenpairs
\[
(-6, \begin{pmatrix} 0 \\ 1 \end{pmatrix}), \quad (15, \begin{pmatrix} 7 \\ -2 \end{pmatrix}).
\]

Near \((0, 6)\) there is one trajectory that approaches \((0, 6)\) along each side of the invariant line \( x = 0 \). There is also one trajectory that emerges from \((0, 6)\) tangent to each side of the line \( y = 6 - \frac{2}{3}x \).

The stationary point \((1, 0)\) is a *saddle*. The coefficient matrix \(A\) has eigenvalues \(-3\) and 5. Because
\[
A + 3I = \begin{pmatrix} 0 & 2 \\ 0 & 8 \end{pmatrix}, \quad A - 5I = \begin{pmatrix} -8 & 2 \\ 0 & 0 \end{pmatrix},
\]

it has the eigenpairs
\[
(-3, \begin{pmatrix} 1 \\ 0 \end{pmatrix}), \quad (5, \begin{pmatrix} 1 \\ 4 \end{pmatrix}).
\]

Near \((1, 0)\) there is one trajectory that emerges from \((1, 0)\) along each side of the invariant line \( y = 0 \). There is also one trajectory that approaches \((1, 0)\) tangent to each side of the line \( y = 4(x - 1) \).

Finally, the stationary point \((3, 3)\) is a *clockwise spiral sink*. All trajectories in the positive quadrant will spiral into it. A global portrait was sketched during the review session.