

## Solutions to Sample Final Exam Problems, Math 246, Fall 2008

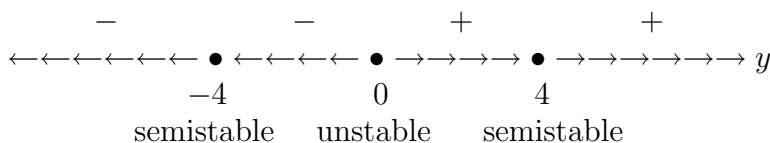
(1) Consider the differential equation  $\frac{dy}{dt} = (16 - y^2)^2 y$  over the interval  $-5 \leq y \leq 5$ .

(a) Identify its equilibrium (stationary) points and classify their stability.

(b) Sketch its phase-line portrait in this interval.

(c) If  $y(0) = 3$ , how does the solution  $y(t)$  behave as  $t \rightarrow \infty$ ?

**Solution (a,b):** The right-hand side factors as  $(4 + y)^2(4 - y)^2 y$ . The stationary solutions are  $y = -4$ ,  $y = 0$ , and  $y = 4$ . A sign analysis of  $(4 + y)^2(4 - y)^2 y$  shows that the phase-line portrait for this equation is therefore



**Solution (c):** The phase-line shows that if  $y(0) = 3$  then  $y(t) \rightarrow 4$  as  $t \rightarrow \infty$ .

(2) Solve (possibly implicitly) each of the following initial-value problems. Identify their intervals of definition.

(a)  $\frac{dy}{dt} + \frac{2ty}{1+t^2} = t^2$ ,  $y(0) = 1$ .

**Solution:** This equation is *linear* and is already in normal form. An integrating factor is

$$\exp\left(\int_0^t \frac{2s}{1+s^2} ds\right) = \exp(\log(1+t^2)) = 1+t^2,$$

so that

$$\frac{d}{dt}((1+t^2)y) = (1+t^2)t^2 = t^2 + t^4.$$

Integrate this to obtain

$$(1+t^2)y = \frac{1}{3}t^3 + \frac{1}{5}t^5 + c.$$

The initial condition  $y(0) = 1$  implies that  $c = (1+0^2) \cdot 1 - \frac{1}{3}0^3 - \frac{1}{5}0^5 = 1$ . Therefore

$$y = \frac{1 + \frac{1}{3}t^3 + \frac{1}{5}t^5}{1+t^2}.$$

This solution exists for every  $t$ .

(b)  $\frac{dy}{dx} + \frac{e^x y + 2x}{2y + e^x} = 0$ ,  $y(0) = 0$ .

**Solution:** Express this equation in the differential form

$$(e^x y + 2x) dx + (2y + e^x) dy = 0.$$

This differential form is *exact* because

$$\partial_y(e^x y + 2x) = e^x = \partial_x(2y + e^x) = e^x.$$

We can therefore find  $H(x, y)$  such that

$$\partial_x H(x, y) = e^x y + 2x, \quad \partial_y H(x, y) = 2y + e^x.$$

The first equation implies  $H(x, y) = e^x y + x^2 + h(y)$ . Plugging this into the second equation gives

$$e^x + h'(y) = 2y + e^x,$$

which yields  $h'(y) = 2y$ . Taking  $h(y) = y^2$ , the general solution is

$$e^x y + x^2 + y^2 = c.$$

The initial condition  $y(0) = 0$  implies that  $c = e^0 \cdot 0 + 0^2 + 0^2 = 0$ . Therefore

$$y^2 + e^x y + x^2 = 0.$$

If you had been asked for an explicit solution then the quadratic formula yields

$$y = \frac{-e^x + \sqrt{e^{2x} - 4x^2}}{2}.$$

Here the positive square root is taken because that solution satisfies the initial condition. It exists wherever  $e^{2x} \geq 4x^2$ .

(3) Let  $y(t)$  be the solution of the initial-value problem

$$\frac{dy}{dt} = y^2 + t^2, \quad y(0) = 1.$$

Use two steps of the forward Euler method to approximate  $y(0.2)$ .

**Solution.** The forward Euler method is

$$\begin{aligned} f_n &= f(y_n, t_n), \\ y_{n+1} &= y_n + hf_n, \\ t_{n+1} &= t_n + h, \end{aligned}$$

where  $h$  is the time step,  $t_0$  is the initial time, and  $y_0$  is the initial data.

When the forward Euler method is applied with  $h = 0.1$ ,  $t_0 = 0$ ,  $y_0 = 1$ , and  $f(y, t) = y^2 + t^2$  for two steps

$$\begin{aligned} f_0 &= f(y_0, t_0) = y_0^2 + t_0^2 = 1^2 + 0^2 = 1, \\ y_1 &= y_0 + hf_0 = 1 + 0.1 \cdot 1 = 1.1, \\ t_1 &= t_0 + h = 0 + 0.1 = 0.1, \\ f_1 &= f(y_1, t_1) = y_1^2 + t_1^2 = (1.1)^2 + (0.1)^2, \\ y_2 &= y_1 + hf_1 = 1.1 + 0.1 \cdot ((1.1)^2 + (0.1)^2). \end{aligned}$$

The approximation is therefore

$$y(0.2) \approx 1.1 + 0.1 \cdot ((1.1)^2 + (0.1)^2).$$

You DO NOT have to work out the arithmetic! If you did then  $y_2 = 1.222$ .

**Remark.** You should be able to answer similar questions that ask you to use the improved Euler (trapezoidal Heun) method.

(4) Give an explicit real-valued general solution of the following equations.

(a)  $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 5y = te^t + \cos(2t)$

**Solution.** This is a constant coefficient, inhomogeneous, linear equation. Its characteristic polynomial is

$$p(z) = z^2 - 2z + 5 = (z - 1)^2 + 4 = (z - 1)^2 + 2^2.$$

This has the conjugate pair of roots  $1 \pm i2$ , which yields a general solution of the associated homogeneous problem

$$y_H(t) = c_1 e^t \cos(2t) + c_2 e^t \sin(2t).$$

A particular solution  $y_P(t)$  can be found by the method of undetermined coefficients. The characteristics of the forcing terms  $te^t$  and  $\cos(2t)$  are  $r + is = 1$  and  $r + is = i2$  respectively. Because these characteristics are different, they should be treated separately. This can be done using either KEY identity evaluation or direct substitution.

**KEY Identity Evaluations.** The forcing term  $te^t$  has degree  $d = 1$  and characteristic  $r + is = 1$ , which is a root of  $p(z)$  of multiplicity  $m = 0$ . Because  $m + d = 1$ , you need the KEY identity and its first derivative

$$\begin{aligned} L(e^{zt}) &= (z^2 - 2z + 5)e^{zt}, \\ L(te^{zt}) &= (z^2 - 2z + 5)t e^{zt} + (2z - 2)e^{zt}. \end{aligned}$$

Evaluate these at  $z = 1$  to find  $L(e^t) = 4e^t$  and  $L(te^t) = 4te^t$ . Dividing the second of these equations by 4 yields  $L(\frac{1}{4}te^t) = te^t$ , which implies  $y_{P1}(t) = \frac{1}{4}te^t$ . The forcing term  $\cos(2t)$  has degree  $d = 0$  and characteristic  $r + is = i2$ , which is a root of  $p(z)$  of multiplicity  $m = 0$ . Because  $m + d = 0$ , you only need the KEY identity

$$L(e^{zt}) = (z^2 - 2z + 5)e^{zt}.$$

Evaluate this at  $z = i2$  to find  $L(e^{i2t}) = (1 - i4)e^{i2t}$ . Dividing this by  $(1 - i4)$  yields

$$L\left(\frac{e^{i2t}}{1 - i4}\right) = e^{i2t}.$$

Because  $\cos(2t) = \operatorname{Re}(e^{i2t})$ , the above equation implies

$$\begin{aligned} y_{P2}(t) &= \operatorname{Re}\left(\frac{e^{i2t}}{1 - i4}\right) = \operatorname{Re}\left(\frac{(1 + i4)e^{i2t}}{1^2 + 4^2}\right) \\ &= \frac{1}{17} \operatorname{Re}((1 + i4)e^{i2t}) = \frac{1}{17}(\cos(2t) - 4\sin(2t)). \end{aligned}$$

Combining these particular solutions with the general solution of the associated homogeneous problem yields the general solution

$$\begin{aligned} y &= y_H(t) + y_{P1}(t) + y_{P2}(t) \\ &= c_1 e^t \cos(2t) + c_2 e^t \sin(2t) + \frac{1}{4}te^t + \frac{1}{17}\cos(2t) - \frac{4}{17}\sin(2t). \end{aligned}$$

**Direct Substitution.** The forcing term  $t e^t$  has degree  $d = 1$  and characteristic  $r + is = 1$ , which is a root of  $p(z)$  of multiplicity  $m = 0$ . Because  $m = 0$  and  $m + d = 1$ , you seek a particular solution of the form

$$y_{P1}(t) = A_0 t e^t + A_1 e^t.$$

Because

$$y'_{P1}(t) = A_0 t e^t + (A_0 + A_1) e^t, \quad y''_{P1}(t) = A_0 t e^t + (2A_0 + A_1) e^t,$$

one sees that

$$\begin{aligned} \mathbb{L}y_{P1}(t) &= y''_{P1}(t) - 2y'_{P1}(t) + 5y_{P1}(t) \\ &= (A_0 t e^t + (2A_0 + A_1) e^t) - 2(A_0 t e^t + (A_0 + A_1) e^t) \\ &\quad + 5(A_0 t e^t + A_1 e^t) \\ &= 4A_0 t e^t + 4A_1 e^t. \end{aligned}$$

Setting  $4A_0 t e^t + 4A_1 e^t = t e^t$ , we see that  $4A_0 = 1$  and  $4A_1 = 0$ , whereby  $A_0 = \frac{1}{4}$  and  $A_1 = 0$ . Hence,  $y_P(t) = \frac{1}{4} t e^t$ .

The forcing term  $\cos(2t)$  has degree  $d = 0$  and characteristic  $r + is = i2$ , which is a root of  $p(z)$  of multiplicity  $m = 0$ . Because  $m = 0$  and  $m + d = 0$ , you seek a particular solution of the form

$$y_{P2}(t) = A \cos(2t) + B \sin(2t).$$

Because

$$\begin{aligned} y'_{P2}(t) &= -2A \sin(2t) + 2B \cos(2t), \\ y''_{P2}(t) &= -4A \cos(2t) - 4B \sin(2t), \end{aligned}$$

one sees that

$$\begin{aligned} \mathbb{L}y_{P2}(t) &= y''_{P2}(t) - 2y'_{P2}(t) + 5y_{P2}(t) \\ &= (-4A \cos(2t) - 4B \sin(2t)) - 2(-2A \sin(2t) + 2B \cos(2t)) \\ &\quad + 5(A \cos(2t) + B \sin(2t)) \\ &= (A - 4B) \cos(2t) + (B + 4A) \sin(2t). \end{aligned}$$

Setting  $(A - 4B) \cos(2t) + (B + 4A) \sin(2t) = \cos(2t)$ , we see that

$$A - 4B = 1, \quad B + 4A = 0.$$

This system can be solved by any method you choose to find  $A = \frac{1}{17}$  and  $B = -\frac{4}{17}$ , whereby

$$y_{P2}(t) = \frac{1}{17} \cos(2t) - \frac{4}{17} \sin(2t).$$

Combining these particular solutions with the general solution of the associated homogeneous problem yields the general solution

$$\begin{aligned} y &= y_H(t) + y_{P1}(t) + y_{P2}(t) \\ &= c_1 e^t \cos(2t) + c_2 e^t \sin(2t) + \frac{1}{4} t e^t + \frac{1}{17} \cos(2t) - \frac{4}{17} \sin(2t). \end{aligned}$$

(b)  $\frac{d^2y}{dt^2} + 9y = \tan(3t)$

**Solution.** This is a constant coefficient, inhomogeneous, linear equation. Its characteristic polynomial is

$$p(z) = z^2 + 9 = z^2 + 3^2.$$

This has the conjugate pair of roots  $\pm i3$ , which yields a general solution of the associated homogeneous problem

$$y_H(t) = c_1 \cos(3t) + c_2 \sin(3t).$$

Because of the form of the forcing term, you must use either the Green function method or variation of parameters to find a particular solution.

**Variation of Parameters.** The equation is already in normal form. Seek a solution in the form

$$y(t) = u_1(t) \cos(3t) + u_2(t) \sin(3t),$$

where  $u_1'(t)$  and  $u_2'(t)$  satisfy

$$\begin{aligned} u_1'(t) \cos(3t) + u_2'(t) \sin(3t) &= 0, \\ -u_1'(t)3 \sin(3t) + u_2'(t)3 \cos(3t) &= \tan(3t). \end{aligned}$$

Solve this system to find

$$u_1'(t) = -\frac{\sin(3t)^2}{3 \cos(3t)} = \frac{1}{3} \cos(3t) - \frac{1}{3} \sec(3t), \quad u_2'(t) = \frac{1}{3} \sin(3t).$$

Integrate these to find

$$u_1(t) = c_1 + \frac{1}{9} \sin(3t) - \frac{1}{9} \log(\tan(3t) + \sec(3t)), \quad u_2(t) = c_2 - \frac{1}{9} \cos(3t).$$

A general solution is therefore

$$\begin{aligned} y &= u_1(t) \cos(3t) + u_2(t) \sin(3t) \\ &= c_1 \cos(3t) + c_2 \sin(3t) - \frac{1}{9} \cos(3t) \log(\tan(3t) + \sec(3t)). \end{aligned}$$

**Green Function Method.** The associated Green function  $g(t)$  satisfies the initial-value problem

$$\frac{d^2g}{dt^2} + 9g = 0, \quad g(0) = 0, \quad g'(0) = 1.$$

Because  $g(t) = c_1 \cos(3t) + c_2 \sin(3t)$ , the first initial condition implies  $c_1 = g(0) = 0$ . Because then  $g'(t) = 3c_2 \cos(3t)$ , the second initial condition implies  $3c_2 = g'(0) = 1$ . Hence,

$$g(t) = \frac{1}{3} \sin(3t).$$

A particular solution is then given by

$$\begin{aligned} y_P(t) &= \int_0^t g(t-s) \tan(3s) \, ds = \frac{1}{3} \int_0^t \sin(3t-3s) \tan(3s) \, ds \\ &= \frac{1}{3} \int_0^t (\sin(3t) \cos(3s) - \cos(3t) \sin(3s)) \tan(3s) \, ds \\ &= \frac{1}{3} \sin(3t) \int_0^t \sin(3s) \, ds - \frac{1}{3} \cos(3t) \int_0^t \frac{\sin(3s)^2}{\cos(3s)} \, ds. \end{aligned}$$

Because

$$\begin{aligned} \int_0^t \sin(3s) \, ds &= -\frac{1}{3} \cos(3s) \Big|_{s=0}^t = \frac{1}{3} - \frac{1}{3} \cos(3t), \\ \int_0^t \frac{\sin(3s)^2}{\cos(3s)} \, ds &= \int_0^t \sec(3s) - \cos(3s) \, ds \\ &= \frac{1}{3} \left( \log(\tan(3s) + \sec(3s)) - \sin(3s) \right) \Big|_{s=0}^t \\ &= \frac{1}{3} \left( \log(\tan(3t) + \sec(3t)) - \sin(3t) \right), \end{aligned}$$

you find that

$$\begin{aligned} y_P(t) &= \frac{1}{9} \sin(3t) (1 - \cos(3t)) - \frac{1}{9} \cos(3t) \left( \log(\tan(3t) + \sec(3t)) - \sin(3t) \right) \\ &= \frac{1}{9} \sin(3t) - \frac{1}{9} \cos(3t) \log(\tan(3t) + \sec(3t)). \end{aligned}$$

A general solution is therefore

$$\begin{aligned} y &= y_H(t) + y_P(t) \\ &= c_1 \cos(3t) + c_2 \sin(3t) + \frac{1}{9} \sin(3t) - \frac{1}{9} \cos(3t) \log(\tan(3t) + \sec(3t)). \end{aligned}$$

- (5) When a mass of 2 kilograms is hung vertically from a spring, it stretches the spring 0.5 meters. (Gravitational acceleration is 9.8 m/sec<sup>2</sup>.) At  $t = 0$  the mass is set in motion from 0.3 meters below its equilibrium (rest) position with a upward velocity of 2 m/sec. Neglect drag and assume that the spring force is proportional to its displacement. Formulate an initial-value problem that governs the motion of the mass for  $t > 0$ . (DO NOT solve this initial-value problem; just write it down!)

**Solution.** Let  $h(t)$  be the displacement (in meters) of the mass from its equilibrium (rest) position at time  $t$  (in seconds), with upward displacements being positive. The governing initial-value problem then has the form

$$m \frac{d^2 h}{dt^2} + kh = 0, \quad h(0) = -.3, \quad h'(0) = 2,$$

where  $m$  is the mass and  $k$  is the spring constant. The problem says that  $m = 2$  kilograms. The spring constant is obtained by balancing the weight of the mass ( $mg = 2 \cdot 9.8$  Newtons) with the force applied by the spring when it is stretched .5 m. This gives  $k \cdot 5 = 2 \cdot 9.8$ , or

$$k = \frac{2 \cdot 9.8}{.5} = 4 \cdot 9.8 \text{ Newtons/m.}$$

The governing initial-value problem is therefore

$$2\frac{d^2h}{dt^2} + 4 \cdot 9.8h = 0, \quad h(0) = -.3, \quad h'(0) = 2.$$

Had you chosen positive  $h$  to be downward displacements then the only thing that would differ is the sign of the initial data.

- (6) Give an explicit general solution of the equation

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = 0.$$

Sketch a typical solution for  $t \geq 0$ . If this equation governs a damped spring-mass system, is the system over, under, or critically damped?

**Solution.** This is a constant coefficient, homogeneous, linear equation. Its characteristic polynomial is

$$p(z) = z^2 + 2z + 5 = (z + 1)^2 + 2^2.$$

This has the conjugate pair of roots  $-1 \pm i2$ , which yields a general solution

$$y = c_1e^{-t} \cos(2t) + c_2e^{-t} \sin(2t).$$

When  $c_1^2 + c_2^2 > 0$  this can be put into the amplitude-phase form

$$y = Ae^{-t} \cos(2t - \delta),$$

where  $A > 0$  and  $0 \leq \delta < 2\pi$  are determined from  $c_1$  and  $c_2$  by

$$A = \sqrt{c_1^2 + c_2^2}, \quad \cos(\delta) = \frac{c_1}{A}, \quad \sin(\delta) = \frac{c_2}{A}.$$

In other words,  $(A, \delta)$  are the polar coordinates for the point in the plane whose Cartesian coordinates are  $(c_1, c_2)$ . The sketch should show a decaying oscillation with amplitude  $Ae^{-t}$  and quasiperiod  $\frac{2\pi}{2} = \pi$ . The equation governs an *under damped* spring-mass system because its characteristic polynomial has a conjugate pair of roots.

- (7) Find the Laplace transform  $Y(s)$  of the solution  $y(t)$  to the initial-value problem

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 8y = f(t), \quad y(0) = 2, \quad y'(0) = 4.$$

where

$$f(t) = \begin{cases} 4 & \text{for } 0 \leq t < 2, \\ t^2 & \text{for } 2 \leq t. \end{cases}$$

You may refer to the table in Section 6.2 of the book. (DO NOT take the inverse Laplace transform to find  $y(t)$ ; just solve for  $Y(s)$ !)

**Solution.** The Laplace transform of the initial-value problem is

$$\mathcal{L}[y''](s) + 4\mathcal{L}[y'](s) + 8\mathcal{L}[y](s) = \mathcal{L}[f](s),$$

where

$$\begin{aligned} \mathcal{L}[y](s) &= Y(s), \\ \mathcal{L}[y'](s) &= sY(s) - y(0) = sY(s) - 2, \\ \mathcal{L}[y''](s) &= s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - 2s - 4. \end{aligned}$$

To compute  $\mathcal{L}[f](s)$ , first write  $f$  as

$$\begin{aligned} f(t) &= (1 - u(t-2))4 + u(t-2)t^2 = 4 - u(t-2)4 + u(t-2)t^2 \\ &= 4 + u(t-2)(t^2 - 4) = 4 + u(t-2)((2 + (t-2))^2 - 4) \\ &= 4 + u(t-2)(4(t-2) + (t-2)^2). \end{aligned}$$

Referring to the table of Laplace transforms in the book, item 13 with  $c = 2$ , item 1, and item 3 with  $n = 1$  and  $n = 2$  then show that

$$\begin{aligned} \mathcal{L}[f](s) &= 4\mathcal{L}[1](s) + 4\mathcal{L}[u(t-2)(t-2)](s) + \mathcal{L}[u(t-2)(t-2)^2](s) \\ &= 4\mathcal{L}[1](s) + 4e^{-2s}\mathcal{L}[t](s) + e^{-2s}\mathcal{L}[t^2](s) \\ &= 4\frac{1}{s} + 4e^{-2s}\frac{1}{s^2} + e^{-2s}\frac{2}{s^3}. \end{aligned}$$

The Laplace transform of the initial-value problem then becomes

$$(s^2Y(s) - 2s - 4) + 4(sY(s) - 2) + 8Y(s) = \frac{4}{s} + e^{-2s}\frac{4}{s^2} + e^{-2s}\frac{2}{s^3},$$

which becomes

$$(s^2 + 4s + 8)Y(s) - 2s - 12 = \frac{4}{s} + e^{-2s}\frac{4}{s^2} + e^{-2s}\frac{2}{s^3}.$$

Hence,  $Y(s)$  is given by

$$Y(s) = \frac{1}{s^2 + 4s + 8} \left( 2s + 12 + \frac{4}{s} + e^{-2s}\frac{4}{s^2} + e^{-2s}\frac{2}{s^3} \right).$$

(8) Find the function  $y(t)$  whose Laplace transform  $Y(s)$  is given by

$$(a) \quad Y(s) = \frac{e^{-3s}4}{s^2 - 6s + 5}, \quad (b) \quad Y(s) = \frac{e^{-2s}s}{s^2 + 4s + 8}.$$

You may refer to the table in Section 6.2 of the book.

**Solution (a).** The denominator factors as  $(s-5)(s-1)$ , so the partial fraction decomposition is

$$\frac{4}{s^2 - 6s + 5} = \frac{4}{(s-5)(s-1)} = \frac{1}{s-5} - \frac{1}{s-1}.$$

Referring to the table of Laplace transforms in the book, item 11 with  $n = 0$  and  $a = 5$ , and with  $n = 0$  and  $a = 1$  gives

$$\mathcal{L}[e^{5t}](s) = \frac{1}{s-5}, \quad \mathcal{L}[e^t](s) = \frac{1}{s-1},$$

whereby

$$\frac{4}{s^2 - 6s + 5} = \mathcal{L}[e^{5t}](s) - \mathcal{L}[e^t](s) = \mathcal{L}[e^{5t} - e^t](s).$$

It follows from item 13 with  $c = 3$  and  $f(t) = e^{5t} - e^t$  that

$$\mathcal{L}[u(t-3)(e^{5(t-3)} - e^{t-3})](s) = e^{-3s}\frac{4}{s^2 - 6s + 5} = Y(s).$$



You therefore conclude that

$$y(t) = \mathcal{L}^{-1}[Y(s)](t) = u(t-3)(e^{5(t-3)} - e^{t-3}).$$

**Solution (b).** The denominator does not have real factors. The partial fraction decomposition is

$$\frac{s}{s^2 + 4s + 8} = \frac{s}{(s+2)^2 + 4} = \frac{s+2}{(s+2)^2 + 2^2} - \frac{2}{(s+2)^2 + 2^2}.$$

Referring to the table of Laplace transforms in the book, items 10 and 9 with  $a = -2$  and  $b = 2$  give

$$\mathcal{L}[e^{-2t} \cos(2t)](s) = \frac{s+2}{(s+2)^2 + 2^2}, \quad \mathcal{L}[e^{-2t} \sin(2t)](s) = \frac{2}{(s+2)^2 + 2^2},$$

whereby

$$\begin{aligned} \frac{s}{s^2 + 4s + 8} &= \mathcal{L}[e^{-2t} \cos(2t)](s) - \mathcal{L}[e^{-2t} \sin(2t)](s) \\ &= \mathcal{L}[e^{-2t} (\cos(2t) - \sin(2t))](s). \end{aligned}$$

It follows from item 13 with  $c = 2$  and  $f(t) = e^{-2t} (\cos(2t) - \sin(2t))$  that

$$\mathcal{L}[u(t-2)e^{-2(t-2)} (\cos(2(t-2)) - \sin(2(t-2)))](s) = e^{-2s} \frac{s}{s^2 + 4s + 8} = Y(s).$$

You therefore conclude that

$$y(t) = \mathcal{L}^{-1}[Y(s)](t) = u(t-2)e^{-2(t-2)} (\cos(2(t-2)) - \sin(2(t-2))).$$

(9) Consider the real vector-valued functions  $\mathbf{x}_1(t) = \begin{pmatrix} 1 \\ t \end{pmatrix}$ ,  $\mathbf{x}_2(t) = \begin{pmatrix} t^3 \\ 3 + t^4 \end{pmatrix}$ .

(a) Compute the Wronskian  $W[\mathbf{x}_1, \mathbf{x}_2](t)$ .

**Solution.** The Wronskian is given by

$$W[\mathbf{x}_1, \mathbf{x}_2](t) = \det \begin{pmatrix} 1 & t^3 \\ t & 3 + t^4 \end{pmatrix} = 1 \cdot (3 + t^4) - t \cdot t^3 = 3 + t^4 - t^4 = 3.$$

(b) Find  $\mathbf{A}(t)$  such that  $\mathbf{x}_1, \mathbf{x}_2$  is a fundamental set of solutions to the linear system

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}(t)\mathbf{x}.$$

**Solution.** Set

$$\mathbf{\Psi}(t) = (\mathbf{x}_1(t) \quad \mathbf{x}_2(t)) = \begin{pmatrix} 1 & t^3 \\ t & 3 + t^4 \end{pmatrix}.$$

If  $\mathbf{x}_1$  and  $\mathbf{x}_2$  are solutions to the linear system then the matrix-valued function  $\mathbf{\Psi}$  must satisfy

$$\frac{d\mathbf{\Psi}}{dt}(t) = \mathbf{A}(t)\mathbf{\Psi}(t).$$

Because  $\det(\Psi(t)) = W[\mathbf{x}_1, \mathbf{x}_2](t) = 3 \neq 0$ , we see that  $\Psi(t)$  is a fundamental matrix of the linear system with  $\mathbf{A}(t)$  given by

$$\begin{aligned}\mathbf{A}(t) &= \frac{d\Psi}{dt}(t) \Psi(t)^{-1} = \begin{pmatrix} 0 & 3t^2 \\ 1 & 4t^3 \end{pmatrix} \frac{1}{3} \begin{pmatrix} 3+t^4 & -t^3 \\ -t & 1 \end{pmatrix} \\ &= \frac{1}{3} \begin{pmatrix} -3t^3 & 3t^2 \\ 3-3t^3 & 3t^3 \end{pmatrix} = \begin{pmatrix} -t^3 & t^2 \\ 1-t^3 & t^3 \end{pmatrix}.\end{aligned}$$

It follows that  $\mathbf{x}_1, \mathbf{x}_2$  is a fundamental set of solutions to the linear system with this  $\mathbf{A}(t)$ .

(c) Give a general solution to the system you found in part (b).

**Solution.** Because  $\mathbf{x}_1, \mathbf{x}_2$  is a fundamental set of solutions to the linear system with the above  $\mathbf{A}(t)$ , a general solution is given by

$$\mathbf{x}(t) = c_1 \mathbf{x}_1 + c_2 \mathbf{x}_2 = c_1 \begin{pmatrix} 1 \\ t \end{pmatrix} + c_2 \begin{pmatrix} t^3 \\ 3+t^4 \end{pmatrix}.$$

(10) Give a general real vector-valued solution of the linear planar system  $\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$  for

$$(a) \quad \mathbf{A} = \begin{pmatrix} 6 & 4 \\ 4 & 0 \end{pmatrix}, \quad (b) \quad \mathbf{A} = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}.$$

**Solution (a).** The characteristic polynomial of  $\mathbf{A}$  is

$$\begin{aligned}p(z) &= z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) \\ &= z^2 - 6z - 16 = (z-3)^2 - 25 = (z-3)^2 - 5^2.\end{aligned}$$

The eigenvalues of  $\mathbf{A}$  are the roots of this polynomial, which are  $3 \pm 5$ , or simply  $-2$  and  $8$ . One therefore has

$$\begin{aligned}e^{t\mathbf{A}} &= e^{3t} \left[ \mathbf{I} \cosh(5t) + (\mathbf{A} - 3\mathbf{I}) \frac{\sinh(5t)}{5} \right] \\ &= e^{3t} \left[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cosh(5t) + \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix} \frac{\sinh(5t)}{5} \right] \\ &= e^{3t} \begin{pmatrix} \cosh(5t) + \frac{3}{5} \sinh(5t) & \frac{4}{5} \sinh(5t) \\ \frac{4}{5} \sinh(5t) & \cosh(5t) - \frac{3}{5} \sinh(5t) \end{pmatrix}.\end{aligned}$$

A general solution is therefore given by

$$\mathbf{x} = c_1 e^{3t} \begin{pmatrix} \cosh(5t) + \frac{3}{5} \sinh(5t) \\ \frac{4}{5} \sinh(5t) \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} \frac{4}{5} \sinh(5t) \\ \cosh(5t) - \frac{3}{5} \sinh(5t) \end{pmatrix}.$$

**Alternative Solution (a).** The characteristic polynomial of  $\mathbf{A}$  is

$$\begin{aligned}p(z) &= z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) \\ &= z^2 - 6z - 16 = (z-3)^2 - 25 = (z-3)^2 - 5^2.\end{aligned}$$

The eigenvalues of  $\mathbf{A}$  are the roots of this polynomial, which are  $3 \pm 5$ , or simply  $-2$  and  $8$ . Because

$$\mathbf{A} + 2\mathbf{I} = \begin{pmatrix} 8 & 4 \\ 4 & 2 \end{pmatrix}, \quad \mathbf{A} - 8\mathbf{I} = \begin{pmatrix} -2 & 4 \\ 4 & -8 \end{pmatrix},$$

we see that  $\mathbf{A}$  has the eigenpairs

$$\left(-2, \begin{pmatrix} 1 \\ -2 \end{pmatrix}\right), \quad \left(8, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right).$$

A general solution is therefore given by

$$\mathbf{x} = c_1 e^{-2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 e^{8t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

**Solution (b).** The characteristic polynomial of  $\mathbf{A}$  is

$$\begin{aligned} p(z) &= z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) \\ &= z^2 - 2z + 5 = (z - 1)^2 + 4 = (z - 1)^2 + 2^2. \end{aligned}$$

The eigenvalues of  $\mathbf{A}$  are the roots of this polynomial, which are  $1 \pm i2$ . One therefore has

$$\begin{aligned} e^{t\mathbf{A}} &= e^t \left[ \mathbf{I} \cos(2t) + (\mathbf{A} - \mathbf{I}) \frac{\sin(2t)}{2} \right] \\ &= e^t \left[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos(2t) + \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} \frac{\sin(2t)}{2} \right] \\ &= e^t \begin{pmatrix} \cos(2t) & \sin(2t) \\ -\sin(2t) & \cos(2t) \end{pmatrix}. \end{aligned}$$

A general solution is therefore given by

$$\mathbf{x} = c_1 e^t \begin{pmatrix} \cos(2t) \\ -\sin(2t) \end{pmatrix} + c_2 e^t \begin{pmatrix} \sin(2t) \\ \cos(2t) \end{pmatrix}.$$

**Alternative Solution (b).** The characteristic polynomial of  $\mathbf{A}$  is

$$\begin{aligned} p(z) &= z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) \\ &= z^2 - 2z + 5 = (z - 1)^2 + 4 = (z - 1)^2 + 2^2. \end{aligned}$$

The eigenvalues of  $\mathbf{A}$  are the roots of this polynomial, which are  $1 \pm i2$ . Because

$$\mathbf{A} - (1 + i2)\mathbf{I} = \begin{pmatrix} -i2 & 2 \\ -2 & -i2 \end{pmatrix}, \quad \mathbf{A} - (1 - i2)\mathbf{I} = \begin{pmatrix} i2 & 2 \\ -2 & i2 \end{pmatrix},$$

we see that  $\mathbf{A}$  has the eigenpairs

$$\left(1 + i2, \begin{pmatrix} 1 \\ i \end{pmatrix}\right), \quad \left(1 - i2, \begin{pmatrix} -i \\ 1 \end{pmatrix}\right).$$

Because

$$e^{(1+i2)t} \begin{pmatrix} 1 \\ i \end{pmatrix} = e^t \begin{pmatrix} \cos(2t) + i \sin(2t) \\ -\sin(2t) + i \cos(2t) \end{pmatrix},$$

two real solutions of the system are

$$e^t \begin{pmatrix} \cos(2t) \\ -\sin(2t) \end{pmatrix}, \quad e^t \begin{pmatrix} \sin(2t) \\ \cos(2t) \end{pmatrix}.$$

A general solution is therefore

$$\mathbf{x} = c_1 e^t \begin{pmatrix} \cos(2t) \\ -\sin(2t) \end{pmatrix} + c_2 e^t \begin{pmatrix} \sin(2t) \\ \cos(2t) \end{pmatrix}.$$

(11) A real  $2 \times 2$  matrix  $\mathbf{A}$  has eigenvalues 2 and  $-1$  with associated eigenvectors

$$\begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

(a) Give a general solution to the linear planar system  $\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$ .

**Solution.** A general solution is

$$\mathbf{x} = c_1 e^{2t} \begin{pmatrix} 3 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

(b) Classify the stability of the origin. Sketch a phase-plane portrait for this system and identify its type. (Carefully mark all sketched trajectories with arrows!)

**Solution.** The coefficient matrix has two real eigenvalues of opposite sign. The origin is therefore a *saddle* and is thereby *unstable*. There is one trajectory moves away from  $(0, 0)$  along each half of the line  $x = 3y$ , and one trajectory moves towards  $(0, 0)$  along each half of the line  $y = -2x$ . (These are the lines of eigenvectors.) Every other trajectory sweeps away from the line  $y = -2x$  and towards the line  $x = 3y$ . A phase-plane portrait was sketched during the review session.

(12) Consider the nonlinear planar system

$$\begin{aligned} \frac{dx}{dt} &= -5y, \\ \frac{dy}{dt} &= x - 4y - x^2. \end{aligned}$$

(a) Find all of its equilibrium (critical, stationary) points.

**Solution.** Stationary points satisfy

$$0 = -5y, \quad 0 = x - 4y - x^2.$$

The first equation implies  $y = 0$ , whereby the second equation becomes  $0 = x - x^2 = x(1 - x)$ , which implies either  $x = 0$  or  $x = 1$ . All the stationary points of the system are therefore

$$(0, 0), \quad (1, 0).$$

- (b) Compute the coefficient matrix of the linearization (the derivative matrix) at each equilibrium (critical, stationary) point.

**Solution.** Because

$$\begin{pmatrix} f(x, y) \\ g(x, y) \end{pmatrix} = \begin{pmatrix} -5y \\ x - 4y - x^2 \end{pmatrix},$$

the matrix of partial derivatives is

$$\begin{pmatrix} \partial_x f(x, y) & \partial_y f(x, y) \\ \partial_x g(x, y) & \partial_y g(x, y) \end{pmatrix} = \begin{pmatrix} 0 & -5 \\ 1 - 2x & -4 \end{pmatrix}.$$

Evaluating this matrix at each stationary point yields the coefficient matrices

$$\mathbf{A} = \begin{pmatrix} 0 & -5 \\ 1 & -4 \end{pmatrix} \quad \text{at } (0, 0), \quad \mathbf{A} = \begin{pmatrix} 0 & -5 \\ -1 & -4 \end{pmatrix} \quad \text{at } (1, 0).$$

- (c) Classify the type and stability of each equilibrium (critical, stationary) point.

**Solution.** The coefficient matrix  $\mathbf{A}$  at  $(0, 0)$  has eigenvalues that satisfy

$$0 = \det(z\mathbf{I} - \mathbf{A}) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 + 4z + 5 = (z + 2)^2 + 1^2.$$

The eigenvalues are thereby  $-2 \pm i$ . Because  $a_{21} = 1 > 0$ , the stationary point  $(0, 0)$  is therefore a *counterclockwise spiral sink*, which is *asymptotically stable* or *attracting*. This is one of the generic types, so it describes the phase-plane portrait of the nonlinear system near  $(0, 0)$ .

The coefficient matrix  $\mathbf{A}$  at  $(1, 0)$  has eigenvalues that satisfy

$$0 = \det(z\mathbf{I} - \mathbf{A}) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 + 4z - 5 = (z + 2)^2 - 3^2.$$

The eigenvalues are thereby  $-2 \pm 3$ , or simply  $-5$  and  $1$ . The stationary point  $(1, 0)$  is therefore a *saddle*, which is *unstable*. This is one of the generic types, so it describes the phase-plane portrait of the nonlinear system near  $(1, 0)$ .

- (d) Sketch a plausible global phase-plane portrait. (Carefully mark all sketched trajectories with arrows!)

**Solution.** The nullcline for  $\frac{dx}{dt}$  is the line  $y = 0$ . This line partitions the plane into regions where  $x$  is increasing or decreasing as  $t$  increases. The nullcline for  $\frac{dy}{dt}$  is the parabola  $y = \frac{1}{4}(x - x^2)$ . This curve partitions the plane into regions where  $y$  is increasing or decreasing as  $t$  increases. Neither of these nullclines is invariant.

The stationary point  $(0, 0)$  is a *counterclockwise spiral sink*.

The stationary point  $(1, 0)$  is a *saddle*. The coefficient matrix  $\mathbf{A}$  has eigenvalues  $-5$  and  $1$ . Because

$$\mathbf{A} + 5\mathbf{I} = \begin{pmatrix} 5 & -5 \\ -1 & 1 \end{pmatrix}, \quad \mathbf{A} - \mathbf{I} = \begin{pmatrix} -1 & -5 \\ -1 & -5 \end{pmatrix},$$

it has the eigenpairs

$$\left(-5, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right), \quad \left(1, \begin{pmatrix} -5 \\ 1 \end{pmatrix}\right)$$

Near  $(1, 0)$  there is one trajectory that emerges from  $(1, 0)$  tangent to each side of the line  $x = 1 - 5y$ . There is also one trajectory that approaches  $(1, 0)$  tangent to each side of the line  $y = x - 1$ . These trajectories are separatrices. A global phase-plane portrait was sketched during the review session.

**Remark.** The global phase-plane portrait becomes clearer if you are able to observe that  $H(x, y) = \frac{1}{2}x^2 + \frac{5}{2}y^2 - \frac{1}{3}x^3$  satisfies

$$\begin{aligned}\frac{d}{dt}H(x, y) &= \partial_x H(x, y) \frac{dx}{dt} + \partial_y H(x, y) \frac{dy}{dt} \\ &= (x - x^2)(-5y) + 5y(x - 4y - x^2) = -20y^2 \leq 0.\end{aligned}$$

The trajectories of the system are thereby seen to cross the level sets of  $H(x, y)$  so as to decrease  $H(x, y)$ . You would not be expected to see this on the Final.

(13) Consider the nonlinear planar system

$$\begin{aligned}\frac{dx}{dt} &= x(3 - 3x + 2y), \\ \frac{dy}{dt} &= y(6 - x - y).\end{aligned}$$

Do parts (a-d) as for the previous problem.

(a) Find all of its equilibrium (critical, stationary) points.

**Solution.** Stationary points satisfy

$$0 = x(3 - 3x + 2y), \quad 0 = y(6 - x - y).$$

The first equation implies either  $x = 0$  or  $3 - 3x + 2y = 0$ , while the second equation implies either  $y = 0$  or  $6 - x - y = 0$ . If  $x = 0$  and  $y = 0$  then  $(0, 0)$  is a stationary point. If  $x = 0$  and  $6 - x - y = 0$  then  $(0, 6)$  is a stationary point. If  $3 - 3x + 2y = 0$  and  $y = 0$  then  $(1, 0)$  is a stationary point. If  $3 - 3x + 2y = 0$  and  $6 - x - y = 0$  then upon solving these equations one finds that  $(3, 3)$  is a stationary point. All the stationary points of the system are therefore

$$(0, 0), \quad (0, 6), \quad (1, 0), \quad (3, 3).$$

(b) Compute the coefficient matrix of the linearization (the derivative matrix) at each equilibrium (critical, stationary) point.

**Solution.** Because

$$\begin{pmatrix} f(x, y) \\ g(x, y) \end{pmatrix} = \begin{pmatrix} 3x - 3x^2 + 2xy \\ 6y - xy - y^2 \end{pmatrix},$$

the matrix of partial derivatives is

$$\begin{pmatrix} \partial_x f(x, y) & \partial_y f(x, y) \\ \partial_x g(x, y) & \partial_y g(x, y) \end{pmatrix} = \begin{pmatrix} 3 - 6x + 2y & 2x \\ -y & 6 - x - 2y \end{pmatrix}.$$

Evaluating this matrix at each stationary point yields the coefficient matrices

$$\begin{aligned}\mathbf{A} &= \begin{pmatrix} 3 & 0 \\ 0 & 6 \end{pmatrix} && \text{at } (0, 0), && \mathbf{A} &= \begin{pmatrix} 15 & 0 \\ -6 & -6 \end{pmatrix} && \text{at } (0, 6), \\ \mathbf{A} &= \begin{pmatrix} -3 & 2 \\ 0 & 5 \end{pmatrix} && \text{at } (1, 0), && \mathbf{A} &= \begin{pmatrix} -9 & 6 \\ -3 & -3 \end{pmatrix} && \text{at } (3, 3).\end{aligned}$$

- (c) Identify the type and stability of each equilibrium (critical, stationary) point.

**Solution.** The coefficient matrix  $\mathbf{A}$  at  $(0, 0)$  is diagonal, so you can read-off its eigenvalues as 3 and 6. The stationary point  $(0, 0)$  is thereby a *nodal source*, which is *unstable* (or even better is *repelling*). This is one of the generic types, so it describes the phase-plane portrait of the nonlinear system near  $(0, 0)$ .

The coefficient matrix  $\mathbf{A}$  at  $(0, 6)$  is triangular, so you can read-off its eigenvalues as  $-6$  and  $15$ . The stationary point  $(0, 6)$  is thereby a *saddle*, which is *unstable*. This is one of the generic types, so it describes the phase-plane portrait of the nonlinear system near  $(0, 6)$ .

The coefficient matrix  $\mathbf{A}$  at  $(1, 0)$  is triangular, so you can read-off its eigenvalues as  $-3$  and  $5$ . The stationary point  $(1, 0)$  is thereby a *saddle*, which is *unstable*. This is one of the generic types, so it describes the phase-plane portrait of the nonlinear system near  $(1, 0)$ .

The coefficient matrix  $\mathbf{A}$  at  $(0, 6)$  has eigenvalues that satisfy

$$0 = \det(z\mathbf{I} - \mathbf{A}) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 + 12z + 45 = (z + 6)^2 + 3^2.$$

Its eigenvalues are thereby  $-6 \pm i3$ . Because  $a_{21} = -3 < 0$ , the stationary point  $(3, 3)$  is therefore a *clockwise spiral sink*, which is *asymptotically stable* or *attracting*. This is one of the generic types, so it describes the phase-plane portrait of the nonlinear system near  $(3, 3)$ .

- (d) Sketch a plausible global phase-plane portrait. (Carefully mark all sketched trajectories with arrows!)

**Solution.** The nullclines for  $\frac{dx}{dt}$  are the lines  $x = 0$  and  $3 - 3x + 2y = 0$ . These lines partition the plane into regions where  $x$  is increasing or decreasing as  $t$  increases. The nullclines for  $\frac{dy}{dt}$  are the lines  $y = 0$  and  $6 - x - y = 0$ . These lines partition the plane into regions where  $y$  is increasing or decreasing as  $t$  increases.

Next, observe that the lines  $x = 0$  and  $y = 0$  are invariant. A trajectory that starts on one of these lines must stay on that line. Along the line  $x = 0$  the system reduces to

$$\frac{dy}{dt} = y(6 - y).$$

Along the line  $y = 0$  the system reduces to

$$\frac{dx}{dt} = 3x(1 - x).$$

The arrows along these invariant lines can be determined from a phase-line portrait of these reduced systems.

The stationary point  $(0, 0)$  is a *nodal source*. The coefficient matrix  $\mathbf{A}$  has eigenvalues 3 and 6. Because

$$\mathbf{A} - 3\mathbf{I} = \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}, \quad \mathbf{A} - 6\mathbf{I} = \begin{pmatrix} -3 & 0 \\ 0 & 0 \end{pmatrix},$$

it has the eigenpairs

$$\left( 3, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right), \quad \left( 6, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$$

Near there is one trajectory that emerges from  $(0, 0)$  along each side of the invariant lines  $y = 0$  and  $x = 0$ . Every other trajectory emerges from  $(0, 0)$  tangent to the line  $y = 0$ , which is the line corresponding to the eigenvalue with the smaller absolute value.

The stationary point  $(0, 6)$  is a *saddle*. The coefficient matrix  $\mathbf{A}$  has eigenvalues  $-6$  and  $15$ . Because

$$\mathbf{A} + 6\mathbf{I} = \begin{pmatrix} 21 & 0 \\ -6 & 0 \end{pmatrix}, \quad \mathbf{A} - 15\mathbf{I} = \begin{pmatrix} 0 & 0 \\ -6 & -21 \end{pmatrix},$$

it has the eigenpairs

$$\left(-6, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right), \quad \left(15, \begin{pmatrix} 7 \\ -2 \end{pmatrix}\right)$$

Near  $(0, 6)$  there is one trajectory that approaches  $(0, 6)$  along each side of the invariant line  $x = 0$ . There is also one trajectory that emerges from  $(0, 6)$  tangent to each side of the line  $y = 6 - \frac{2}{7}x$ . These trajectories are separatrices.

The stationary point  $(1, 0)$  is a *saddle*. The coefficient matrix  $\mathbf{A}$  has eigenvalues  $-3$  and  $5$ . Because

$$\mathbf{A} + 3\mathbf{I} = \begin{pmatrix} 0 & 2 \\ 0 & 8 \end{pmatrix}, \quad \mathbf{A} - 5\mathbf{I} = \begin{pmatrix} -8 & 2 \\ 0 & 0 \end{pmatrix},$$

it has the eigenpairs

$$\left(-3, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right), \quad \left(5, \begin{pmatrix} 1 \\ 4 \end{pmatrix}\right)$$

Near  $(1, 0)$  there is one trajectory that emerges from  $(1, 0)$  along each side of the invariant line  $y = 0$ . There is also one trajectory that approaches  $(1, 0)$  tangent to each side of the line  $y = 4(x - 1)$ . These trajectories are also separatrices.

Finally, the stationary point  $(3, 3)$  is a *clockwise spiral sink*. All trajectories in the positive quadrant will spiral into it. A phase-plane global portrait was sketched during the review session.