Third In-Class Exam Solutions  
Math 246, Spring 2009, Professor David Levermore  
Thursday, 30 April 2009

(1) [8] Consider the matrices
\[ \mathbf{A} = \begin{pmatrix} 2 & -4 \\ 1 & 3 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 3 & 5 \\ 2 & 4 \end{pmatrix}. \]

Compute the matrices

(a) \( \mathbf{AB} \)

\[ \mathbf{AB} = \begin{pmatrix} 2 & -4 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 3 & 5 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} -2 & -6 \\ 9 & 17 \end{pmatrix} \]

(b) \( \mathbf{B}^{-1} \)

\[ \mathbf{B}^{-1} = \frac{1}{\det(\mathbf{B})} \begin{pmatrix} 4 & -5 \\ -2 & 3 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 4 & -5 \\ -2 & 3 \end{pmatrix}. \]

(2) [15] Consider the matrix
\[ \mathbf{A} = \begin{pmatrix} 5 & 2 \\ 8 & -1 \end{pmatrix}. \]

(a) Find all the eigenvalues of \( \mathbf{A} \).

\textbf{Solution.} The characteristic polynomial of \( \mathbf{A} \) is given by \[ p(z) = z^2 - \text{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 4z - 21 = (z + 3)(z - 7). \]

The eigenvalues of \( \mathbf{A} \) are the roots of this polynomial, which are \(-3\) and \(7\).

(b) For each eigenvalue of \( \mathbf{A} \) find an associated eigenvector.

\textbf{Solution (using the Cayley-Hamilton method from notes).} One has \[ \mathbf{A} + 3\mathbf{I} = \begin{pmatrix} 8 & 2 \\ 8 & 2 \end{pmatrix}, \quad \mathbf{A} - 7\mathbf{I} = \begin{pmatrix} -2 & 2 \\ 8 & -8 \end{pmatrix}. \]

Every nonzero column of \( \mathbf{A} - 7\mathbf{I} \) has the form
\[ \alpha_1 \begin{pmatrix} -1 \\ 4 \end{pmatrix} \] for some \( \alpha_1 \neq 0 \), any of which is an eigenvector associated with \(-3\). Similarly, every nonzero column of \( \mathbf{A} + 3\mathbf{I} \) has the form
\[ \alpha_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \] for some \( \alpha_2 \neq 0 \), any of which is an eigenvector associated with \(7\).
(c) Diagonalize A.

**Solution.** Because A has the eigenpairs
\[
\begin{pmatrix} 7, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{pmatrix}, \quad \begin{pmatrix} -3, \begin{pmatrix} -1 \\ 4 \end{pmatrix} \end{pmatrix},
\]
set
\[
V = \begin{pmatrix} 1 & -1 \\ 1 & 4 \end{pmatrix}, \quad D = \begin{pmatrix} 7 & 0 \\ 0 & -3 \end{pmatrix}.
\]
Because \( \det(V) = 4 - (-1) = 5 \),
\[
V^{-1} = \frac{1}{\det(V)} \begin{pmatrix} 4 & 1 \\ -1 & 1 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 4 & 1 \\ -1 & 1 \end{pmatrix}.
\]
Then A has the diagonalization
\[
A = VDV^{-1} = \begin{pmatrix} 1 & -1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 7 & 0 \\ 0 & -3 \end{pmatrix} \frac{1}{5} \begin{pmatrix} 4 & 1 \\ -1 & 1 \end{pmatrix}.
\]

(3) [10] Suppose you know that \( e^{tA} = \begin{pmatrix} \cos(2t) + \sin(2t) & -\sin(2t) \\ 2 \sin(2t) & \cos(2t) - \sin(2t) \end{pmatrix} \).

(a) Solve the initial-value problem
\[
\frac{d}{dt} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = A \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.
\]

**Solution.** The solution is given by
\[
\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{tA} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} \cos(2t) + \sin(2t) & -\sin(2t) \\ 2 \sin(2t) & \cos(2t) - \sin(2t) \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix}
\]
\[
= \begin{pmatrix} \cos(2t) - \sin(2t) \\ 2 \cos(2t) \end{pmatrix}.
\]

(b) Determine A.

**Solution.** The simplest way to do this is
\[
A = \left. \frac{de^{tA}}{dt} \right|_{t=0} = \left. \begin{pmatrix} -2 \sin(2t) + 2 \cos(2t) & -2 \cos(2t) \\ 4 \cos(2t) & -2 \sin(2t) - 2 \cos(2t) \end{pmatrix} \right|_{t=0} = \begin{pmatrix} 2 & -2 \\ 4 & -2 \end{pmatrix}.
\]

**Alternative Solution.** Because \( \frac{de^{tA}}{dt} = Ae^{tA} \), and because \( (e^{tA})^{-1} = e^{-tA} \), you see that
\[
A = \frac{de^{tA}}{dt} \left( e^{tA} \right)^{-1} = \frac{de^{tA}}{dt} e^{-tA}.
\]
Because A is independent of t you may evaluate the right-hand side at any t. It is best to set \( t = 0 \) on the right-hand side because \( e^{0A} = I \). The right-hand side is then evaluated as in the previous solution.
(4) [10] Consider two interconnected tanks filled with brine (salt water). The first tank contains 70 liters and the second contains 40 liters. Brine flows with a concentration of 3 grams of salt per liter flows into the first tank at a rate of 5 liters per hour. Well stirred brine flows from the first tank to the second at a rate of 7 liters per hour, from the second to the first at a rate of 2 liters per hour, and from the second into a drain at a rate of 5 liters per hour. At \( t = 0 \) there are 35 grams of salt in the first tank and 25 grams in the second. Give an initial-value problem that governs the amount of salt in each tank as a function of time.

**Solution.** The rates work out so there will always be 70 liters of brine in the first tank and 40 liters in the second. Let \( S_1(t) \) and \( S_2(t) \) be the grams of salt in the first and second tanks respectively. These are governed by the initial-value problem

\[
\frac{dS_1}{dt} = 3 \cdot 5 + \frac{S_2}{40} \cdot 2 - \frac{S_1}{70} \cdot 7, \quad S_1(0) = 35,
\]

\[
\frac{dS_2}{dt} = \frac{S_1}{70} \cdot 7 - \frac{S_2}{40} \cdot 2 - \frac{S_2}{40} \cdot 5, \quad S_2(0) = 25.
\]

(5) [8] Transform the equation \( \frac{d^4y}{dt^4} + e^t \frac{d^3y}{dt^3} - \frac{dy}{dt} + 5y = t^2 \) into a first-order system of ordinary differential equations.

**Solution:** Because the equation is fourth order, the first order system must have dimension four. The simplest such first order system is

\[
\frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ x_4 \\ t^2 - 5x_1 + x_2 - e^t x_4 \end{pmatrix}, \quad \text{where} \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} y \\ y' \\ y'' \end{pmatrix}.
\]

(6) [15] Consider the vector-valued functions \( x_1(t) = \begin{pmatrix} 1 + t^5 \\ 2t^2 \end{pmatrix}, \quad x_2(t) = \begin{pmatrix} t^3 \\ 2 \end{pmatrix} \).

(a) Compute the Wronskian \( W[x_1, x_2](t) \).

**Solution.**

\[
W[x_1, x_2](t) = \det \begin{pmatrix} 1 + t^5 & t^3 \\ 2t^2 & 2 \end{pmatrix} = (1 + t^5)2 - 2t^5 = 2.
\]

(b) Find \( A(t) \) such that \( x_1, x_2 \) is a fundamental set of solutions to the system \( \frac{dx}{dt} = A(t)x \) wherever \( W[x_1, x_2](t) \neq 0 \).

**Solution.** Let \( \Psi(t) = \begin{pmatrix} 1 + t^5 & t^3 \\ 2t^2 & 2 \end{pmatrix} \). Because \( \frac{d\Psi(t)}{dt} = A(t)\Psi(t) \), one has

\[
A(t) = \frac{\Psi(t)}{\frac{d\Psi(t)}{dt}} \Psi(t)^{-1} = \begin{pmatrix} 5t^4 & 3t^2 \\ 4t & 0 \end{pmatrix} \begin{pmatrix} 1 + t^5 & t^3 \\ 2t^2 & 2 \end{pmatrix}^{-1}
\]

\[
= \begin{pmatrix} 5t^4 & 3t^2 \\ 4t & 0 \end{pmatrix} \frac{1}{2} \begin{pmatrix} 2 & -t^3 \\ -2t^2 & 1 + t^5 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2t^4 & 3t^3 - 2t^7 \\ 4t & -4t^4 \end{pmatrix} = \begin{pmatrix} 2t^4 & \frac{2}{3}t^2 - t^7 \\ 4t & -2t^4 \end{pmatrix}.
\]
(c) Give a general solution to the system you found in part (b).

**Solution.** Because $W[x_1, x_2](t) = 2 \neq 0$, a general solution is

$$x = c_1 x_1(t) + c_2 x_2(t) = c_1 \left(1 + \frac{t^5}{2t^2}\right) + c_2 \left(t^3\right).$$

(7) [16] Find a general solution for each of the following systems.

(a) $\frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 & 4 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

**Solution.** Let $A = \begin{pmatrix} -3 & 4 \\ -1 & 1 \end{pmatrix}$. The characteristic polynomial of $A$ is

$$p(z) = z^2 - \text{tr}(A)z + \det(A) = z^2 + 2z + 1 = (z + 1)^2,$$

which has the double root $-1$. Then, because $\mu = -1$ and $\nu = 0$,

$$e^{tA} = e^{-t} \left[I + (A + I)t\right] = e^{-t} \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -2 & 4 \\ -1 & 2 \end{pmatrix} t\right]$$

$$= e^{-t} \begin{pmatrix} 1 - 2t & 4t \\ -t & 1 + 2t \end{pmatrix}.$$  

A general solution is therefore

$$\begin{pmatrix} x \\ y \end{pmatrix} = e^{tA} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = c_1 e^{-t} \begin{pmatrix} 1 - 2t \\ -t \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 4t \\ 1 + 2t \end{pmatrix}.$$

(b) $\frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 5 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

**Solution.** Let $A = \begin{pmatrix} 1 & -2 \\ 5 & -1 \end{pmatrix}$. The characteristic polynomial of $A$ is

$$p(z) = z^2 - \text{tr}(A)z + \det(A) = z^2 + 9 = z^2 + 3^2,$$

which has the conjugate pair of roots $\pm i 3$. Then, because $\mu = 0$ and $\nu = 4$,

$$e^{tA} = I \cos(3t) + A \frac{\sin(3t)}{3} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos(3t) + \begin{pmatrix} 1 & -2 \\ 5 & -1 \end{pmatrix} \frac{\sin(3t)}{3}$$

$$= \begin{pmatrix} \cos(3t) + \frac{1}{3} \sin(3t) & -\frac{2}{3} \sin(4t) \\ \frac{5}{3} \sin(3t) & \cos(3t) - \frac{1}{3} \sin(3t) \end{pmatrix}.$$  

A general solution is therefore

$$\begin{pmatrix} x \\ y \end{pmatrix} = e^{tA} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = c_1 \begin{pmatrix} \cos(3t) + \frac{1}{3} \sin(3t) \\ \frac{5}{3} \sin(3t) \end{pmatrix} + c_2 \begin{pmatrix} -\frac{2}{3} \sin(3t) \\ \cos(3t) - \frac{1}{3} \sin(3t) \end{pmatrix}.$$
(8) [10] Sketch the phase-plane portrait for each of the two systems in the previous problem. For each portrait identify its type and give a reason why the origin is either attracting, stable, unstable, or repelling.

Solution (a). The coefficient matrix $A$ has the eigenvalue $-1$. Because

$$A + I = \begin{pmatrix} -2 & 4 \\ -1 & 2 \end{pmatrix},$$

it has the eigenpair

$$(-1, \begin{pmatrix} 2 \\ 1 \end{pmatrix}).$$

Because $A \neq -I$, the portrait is a twist sink (improper nodal sink) and is thereby attracting (asymptotically stable). Because $a_{21} = -1 < 0$, the phase portrait is a clockwise twist sink. There is one trajectory that approaches the origin along each half of the line $y = \frac{1}{2}x$. Trajectories above the line $y = \frac{1}{2}x$ will approach the origin tangent to the line $y = \frac{1}{2}x$ from the right. Trajectories below the line $y = \frac{1}{2}x$ will approach the origin tangent to the line $y = \frac{1}{2}x$ from the left.

Solution (b). The coefficient matrix $A$ has the eigenvalues $\pm i3$. The portrait is therefore a center and the origin is thereby stable. Because $a_{21} = 5 > 0$, the phase portrait is a counterclockwise center.

(9) [8] Suppose you know that a $2 \times 2$ matrix $A$ can be diagonalized as $A = VDV^{-1}$ where

$$V = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}, \quad D = \begin{pmatrix} 6 & 0 \\ 0 & -4 \end{pmatrix}.$$

Use this information to compute $e^{tA}$.

Solution. Because $e^{tA} = Ve^{tD}V^{-1}$ with $e^{tD} = \begin{pmatrix} e^{6t} & 0 \\ 0 & e^{-4t} \end{pmatrix}$ and $V^{-1} = \frac{1}{5} \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$,

$$e^{tA} = Ve^{tD}V^{-1} = \frac{1}{5} \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} e^{6t} & 0 \\ 0 & e^{-4t} \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 4e^{6t} + e^{-4t} & 2e^{6t} - 2e^{-4t} \\ 2e^{6t} - 2e^{-4t} & e^{6t} + 4e^{-4t} \end{pmatrix}.$$  

Alternative Solution. Because

$$A = VDV^{-1} = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 6 & 0 \\ 0 & -4 \end{pmatrix} \frac{1}{5} \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 12 & 6 \\ -4 & 8 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 20 & 20 \\ 20 & -10 \end{pmatrix} = \begin{pmatrix} 4 & 4 \\ 4 & -2 \end{pmatrix},$$

and because the eigenvalues of $A$ are $1 \pm 5$, we obtain

$$e^{tA} = e^t \left[ I \cosh(5t) + (A - I) \frac{\sinh(5t)}{5} \right] = e^t \left[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cosh(5t) + \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix} \frac{\sinh(5t)}{5} \right]$$

$$= e^t \begin{pmatrix} \cosh(5t) + \frac{3}{5} \sinh(5t) & \frac{4}{5} \sinh(5t) \\ \frac{4}{5} \sinh(5t) & \cosh(5t) - \frac{3}{5} \sinh(5t) \end{pmatrix}.$$  

This is equivalent to the solution given previously.