(1) [5] Consider the first-order planar system
\[
\frac{dx}{dt} = x(1-y), \quad \frac{dy}{dt} = y(x-1).
\]
Its stationary points are \((0,0)\) and \((1,1)\). The coefficient matrix \(A\) of its linearization at these points is respectively
\[
A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]
The first of these matrices has the real eigenpairs
\[
\left(1, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right), \quad \left(-1, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right),
\]
while the second has the conjugate pair of eigenvalues \(\pm i\). Sketch a plausible global phase-plane portrait of this system. Carefully mark all sketched orbits with arrows.

**Solution:**
The stationary point \((0,0)\) is a saddle, and is therefore unstable, because the eigenvalues of its \(A\) are real, nonzero, and have opposite sign. The \(x\)-axis \((y = 0)\) and \(y\)-axis \((x = 0)\) are invariant. Orbits move away from \((0,0)\) along the \(x\)-axis while orbits move towards \((0,0)\) along the \(y\)-axis.
The stationary point \((1,1)\) is a counterclockwise center, and is therefore stable, because \(a_{21} = 1 > 0\), and because the eigenvalues of its \(A\) are \(\pm i\) while the system is conservative in the first quadrant — a fact seen because its integrals satisfy
\[
\frac{x-1}{x} \, dx - \frac{1-y}{y} \, dy = 0,
\]
which has no singularities when \(x \neq 0\) and \(y \neq 0\). A sketch was given in class.

(2) [5] Consider the first-order planar system
\[
\frac{dx}{dt} = y, \quad \frac{dy}{dt} = 5x + x^2 + 4y.
\]
Its stationary points are \((0,0)\) and \((-5,0)\). Classify the type and stability of each of these stationary points. (You do not have to sketch anything.)

**Solution:** The matrix of partial derivatives is
\[
\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 5 + 2x & 4 \end{pmatrix}.
\]
- At \((0,0)\) the coefficient matrix of its linearization is \(A = \begin{pmatrix} 0 & 1 \\ 5 & 4 \end{pmatrix}\), which has the characteristic polynomial \(p(z) = z^2 - 4z - 5 = (z+1)(z-5)\). The eigenvalues of \(A\) are \(-1\) and \(5\). Because these are real, nonzero, and have opposite sign, the stationary point \((0,0)\) is a saddle and is therefore unstable.
- At \((-5,0)\) the coefficient matrix of its linearization is \(A = \begin{pmatrix} 0 & 1 \\ -5 & 4 \end{pmatrix}\), which has the characteristic polynomial \(p(z) = z^2 - 4z + 5 = (z-2)^2 + 1\). The eigenvalues of \(A\) are the conjugate pair \(2 \pm i\). Because these have positive real part, and because \(a_{21} = -5 < 0\), the stationary point \((-5,0)\) is a clockwise spiral source and is therefore repelling.