Solutions of Sample Problems for First In-Class Exam  
Math 246, Fall 2009, Professor David Levermore

(1) (a) Give the integral being evaluated by the MATLAB command
\[
\text{int}('x/(1+x^4)',x',0,inf).
\]

**Solution:** It is evaluating the definite integral
\[
\int_0^\infty \frac{r}{1+r^4} \, dr.
\]
where you can replace \( r \) by any other variable.

(b) Sketch the graph that you expect would be produced by the following MATLAB commands.

\[
[x, y] = \text{meshgrid}(-5:0.5:5,-5:0.5:5)
\]
\[
\text{contour}(x, y, x^2 + y^2, [1, 9, 25])
\]
\[
\text{axis square}
\]

**Solution:** Your sketch should show both \( x \) and \( y \) axes marked from \(-5\) to \(5\) and circles of radius 1, 3, and 5 centered at the origin. The tick marks on the axes should mark intervals of length .5.

(2) Find the explicit solution for each of the following initial-value problems and identify its interval of existence (definition).

(a) \[
\frac{dz}{dt} = \frac{\cos(t) - z}{1 + t}, \quad z(0) = 2.
\]

**Solution:** This equation is *linear* in \( z \), so write it in the linear normal form
\[
\frac{dz}{dt} + \frac{z}{1 + t} = \frac{\cos(t)}{1 + t}.
\]
An integrating factor is given by
\[
\exp \left( \int_0^t \frac{1}{1 + s} \, ds \right) = \exp \left( \log(1 + t) \right) = 1 + t,
\]
Upon multiplying the equation by \( 1 + t \), one finds that
\[
\frac{d}{dt} \left( (1 + t)z \right) = \cos(t),
\]
which is then integrated to obtain
\[
(1 + t)z = \sin(t) + c.
\]
The integration constant \( c \) is found through the initial condition \( z(0) = 2 \) by setting \( t = 0 \) and \( z = 0 \), whereby
\[
c = (1 + 0)2 - \sin(0) = 2.
\]
Hence, upon solving explicitly for \( z \), the solution is
\[
z = \frac{2 + \sin(t)}{1 + t}.
\]
The interval of existence for this solution is \( t > -1 \).
(b) \( \frac{du}{dz} = e^u + 1 , \quad u(0) = 0. \)

**Solution:** This equation is *autonomous* (and therefore *separable*), so write it in the separated differential form
\[
\frac{1}{e^u + 1} \, du = dz .
\]

This equation can be integrated to obtain
\[
z = \int \frac{1}{e^u + 1} \, du = \int \frac{e^{-u}}{1 + e^{-u}} \, du = - \log(1 + e^{-u}) + c .
\]

The integration constant \( c \) is found through the initial condition \( u(0) = 0 \) by setting \( z = 0 \) and \( u = 0 \), whereby
\[
c = 0 + \log(1 + e^0) = \log(2) .
\]

Hence, the solution is given implicitly by
\[
z = - \log(1 + e^{-u}) + \log(2) = - \log \left( \frac{1 + e^{-u}}{2} \right) .
\]

This may be solve for \( u \) as follows:
\[
e^{-z} = \frac{1 + e^{-u}}{2} ,
\]
\[
2e^{-z} - 1 = e^{-u} ,
\]
\[
u = - \log(2e^{-z} - 1) .
\]

The interval of existence for this solution is \( z < \log(2) \).

(3) Consider the differential equation
\[
\frac{dy}{dt} = 4y^2 - y^4 .
\]

(a) Find all of its stationary (equilibrium) solutions and classify each as being either stable, unstable, or semistable.

**Solution:** The right-hand side of the equation factors as
\[
4y^2 - y^4 = y^2(4 - y^2) = y^2(2 + y)(2 - y) ,
\]
which implies that \( y = -2, \ y = 0, \) and \( y = 2 \) are all of its stationary solutions.

A sign analysis of \( y^2(2 + y)(2 - y) \) then shows that
\[
\frac{dy}{dt} > 0 \quad \text{when } -2 < y < 0 \text{ or } 0 < y < 2 ,
\]
\[
\frac{dy}{dt} < 0 \quad \text{when } -\infty < y < -2 \text{ or } 2 < y < \infty .
\]

The phase-line for this equation is therefore
\[
\begin{array}{cccccc}
- & + & + & - \\
\begin{array}{ccc}
-2 & 0 & 2 \\
\text{unstable} & \text{semistable} & \text{stable}
\end{array}
\end{array}
\]
(b) If \( y(0) = 1 \), how does the solution \( y(t) \) behave as \( t \to \infty \)?

**Solution:** It is clear from the answer to (a) that

\[
\frac{dy}{dt} > 0 \quad \text{when } 0 < y < 2,
\]

so that \( y(t) \to 2 \) as \( t \to \infty \) if \( y(0) = 1 \).

(c) If \( y(0) = -1 \), how does the solution \( y(t) \) behave as \( t \to \infty \)?

**Solution:** It is clear from the answer to (a) that

\[
\frac{dy}{dt} > 0 \quad \text{when } -2 < y < 0,
\]

so that \( y(t) \to 0 \) as \( t \to \infty \) if \( y(0) = -1 \).

(d) Sketch a graph of \( y \) versus \( t \) showing the direction field and several solution curves. The graph should show all the stationary solutions as well as solution curves above and below each of them. Every value of \( y \) should lie on at least one sketched solution curve.

**Solution:** Will be given during the review session.

(4) A tank initially contains 100 liters of pure water. Beginning at time \( t = 0 \) brine (salt water) with a salt concentration of 2 grams per liter \((\text{g/l})\) flows into the tank at a constant rate of 3 liters per minute \((\text{l/min})\) and the well-stirred mixture flows out of the tank at the same rate. Let \( S(t) \) denote the mass \((\text{g})\) of salt in the tank at time \( t \geq 0 \).

(a) Write down an initial-value problem that governs \( S(t) \).

**Solution:** Because water flows in and out of the tank at the same rate, the tank will contain 100 liters of salt water for every \( t > 0 \). The salt concentration of the water in the tank at time \( t \) will therefore be \( \frac{S(t)}{100} \). Because this is also the concentration of the outflow, \( S(t) \), the mass of salt in the tank at time \( t \), will satisfy

\[
\frac{dS}{dt} = \text{RATE IN} - \text{RATE OUT} = 2 \cdot 3 - \frac{S}{100} \cdot 3 = 6 - \frac{3}{100}S.
\]

Because there is no salt in the tank initially, the initial-value problem that governs \( S(t) \) is

\[
\frac{dS}{dt} = 6 - \frac{3}{100}S, \quad S(0) = 0.
\]

(b) Is \( S(t) \) an increasing or decreasing function of \( t \)? (Give your reasoning.)

**Solution:** One sees from part (a) that

\[
\frac{dS}{dt} = \frac{3}{100}(200 - S) > 0 \quad \text{for } S < 200,
\]

whereby \( S(t) \) is an increasing function of \( t \) that will approach the stationary value of 200 g as \( t \to \infty \).

(c) What is the behavior of \( S(t) \) as \( t \to \infty \)? (Give your reasoning.)

**Solution:** The argument given for part (b) already shows that \( S(t) \) is an increasing function of \( t \) that approaches the stationary value of 200 g as \( t \to \infty \).
(d) Derive an explicit formula for $S(t)$.

**Solution:** The differential equation given in the answer to part (a) is linear, so write it in the form

$$\frac{dS}{dt} + \frac{3}{100}S = 6.$$ 

An integrating factor is $e^{\frac{3}{100}t}$, whereby

$$\frac{d}{dt}(e^{\frac{3}{100}t}S) = 6e^{\frac{3}{100}t}.$$ 

This is the integrated to obtain

$$e^{\frac{3}{100}t}S = 200e^{\frac{3}{100}t} + c.$$ 

The integration constant $c$ is found by setting $t = 0$ and $S = 0$, whereby

$$c = e^{0} - 200 \cdot e^{0} = -200.$$ 

Then solving for $S$ gives

$$S(t) = 200 - 200e^{-\frac{3}{100}t}.$$ 

(5) Suppose you are using the Heun-midpoint method to numerically approximate the solution of an initial-value problem over the time interval $[0, 5]$. By what factor would you expect the error to decrease when you increase the number of time steps taken from 500 to 2000.

**Solution:** The Heun-midpoint method is second order, which means its (global) error scales like $h^2$ where $h$ is the time step. When the number of time steps taken increases from 500 to 2000, the time step $h$ decreases by a factor of 4. The error will therefore decrease (like $h^2$) by a factor of $4^2 = 16$.

(6) Give an implicit general solution to each of the following differential equations.

(a) $\left(\frac{y}{x} + 3x\right) dx + \left(\log(x) - y\right) dy = 0$.

**Solution:** Because

$$\partial_y \left(\frac{y}{x} + 3x\right) = \frac{1}{x} = \partial_x \left(\log(x) - y\right),$$

the equation is exact. You can therefore find $H(x, y)$ such that

$$\partial_x H(x, y) = \frac{y}{x} + 3x, \quad \partial_y H(x, y) = \log(x) - y.$$ 

The first of these equations implies that

$$H(x, y) = y \log(x) + \frac{3}{2}x^2 + h(y).$$

Plugging this into the second equation then shows that

$$\log(x) - y = \partial_y H(x, y) = \log(x) + h'(y).$$

Hence, $h'(y) = -y$, which yields $h(y) = -\frac{1}{2}y^2$. The general solution is therefore governed implicitly by

$$y \log(x) + \frac{3}{2}x^2 - \frac{1}{2}y^2 = c,$$

where $c$ is an arbitrary constant.
(b) \((x^2 + y^3 + 2x) \, dx + 3y^2 \, dy = 0\).

**Solution:** Because
\[
\partial_y(x^2 + y^3 + 2x) = 3y^2 \neq \partial_x(3y^2) = 0,
\]
the equation is *not exact*. Seek an integrating factor \(\mu(x, y)\) such that
\[
\partial_y((x^2 + y^3 + 2x)\mu) = \partial_x(3y^2\mu).
\]
This means that \(\mu\) must satisfy
\[
(x^2 + y^3 + 2x)\partial_y\mu + 3y^2\mu = 3y^2\partial_x\mu.
\]
If you assume that \(\mu\) depends only on \(x\) (so that \(\partial_y\mu = 0\)) then this reduces to
\[
\mu = \partial_x\mu,
\]
which depends only on \(x\). One sees from this that \(\mu = e^x\) is an integrating factor.
This implies that
\[
(x^2 + y^3 + 2x)e^x \, dx + 3y^2e^x \, dy = 0 \quad \text{is exact}.
\]
You can therefore find \(H(x, y)\) such that
\[
\partial_xH(x, y) = (x^2 + y^3 + 2x)e^x, \quad \partial_yH(x, y) = 3y^2e^x.
\]
The second of these equations implies that
\[
H(x, y) = y^3e^x + h(x).
\]
Plugging this into the first equation then yields
\[
(x^2 + y^3 + 2x)e^x = \partial_xH(x, y) = y^3e^x + h'(x).
\]
Hence, \(h\) satisfies
\[
h'(x) = (x^2 + 2x)e^x.
\]
This can be integrated to obtain \(h(x) = x^2e^x\). The general solution is therefore governed implicitly by
\[
(y^3 + x^2)e^x = c,
\]
where \(c\) is an arbitrary constant.

(7) A 2 kilogram (kg) mass initially at rest is dropped in a medium that offers a resistance of \(\frac{v^2}{40}\) newtons (= kg m/sec\(^2\)) where \(v\) is the downward velocity (m/sec) of the mass. The gravitational acceleration is 9.8 m/sec\(^2\).

(a) What is the terminal velocity of the mass?

**Solution:** The terminal velocity is the velocity at which the force of resistance balances that of gravity. This happens when
\[
\frac{1}{40}v^2 = mg = 2 \cdot 9.8.
\]
Upon solving this for \(v\) one obtains
\[
v = \sqrt{40 \cdot 2 \cdot 9.8} \, \text{m/sec} \quad \text{(full marks)}
\]
\[
= \sqrt{4 \cdot 2 \cdot 98} = \sqrt{4 \cdot 2 \cdot 49}
\]
\[
= \sqrt{4^2 \cdot 7^2} = 4 \cdot 7 = 28 \, \text{m/sec}.
\]
(b) Write down an initial-value problem that governs \( v \) as a function of time. (You do not have to solve it!)

**Solution:** The net downward force on the falling mass is the force of gravity minus the force of resistance. By Newton \((ma = F)\), this leads to

\[
m\frac{dv}{dt} = mg - \frac{1}{40}v^2.
\]

Because \(m = 2\) and \(g = 9.8\), and because the mass is initially at rest, this yields the initial-value problem

\[
\frac{dv}{dt} = 9.8 - \frac{1}{40}v^2, \quad v(0) = 0.
\]

(8) Consider the following MATLAB function M-file.

```matlab
function [t,y] = solveit(ti, yi, tf, n)
    h = (tf - ti)/n;
    t = zeros(n + 1, 1);
    y = zeros(n + 1, 1);
    t(1) = ti;
    y(1) = yi;
    for i = 1:n
        z = t(i)^4 + y(i)^2;
        t(i + 1) = t(i) + h;
        y(i + 1) = y(i) + (h/2)*(z + t(i + 1)^4 + (y(i) + h*z)^2);
    end
```

(a) What is the initial-value problem being approximated numerically?

**Solution:** The initial-value problem being approximated is

\[
\frac{dy}{dt} = t^4 + y^2, \quad y(ti) = yi.
\]

(b) What is the numerical method being used?

**Solution:** The Heun-Trapezoidal (improved Euler) method is being used.

(c) What are the output values of \(t(2)\) and \(y(2)\) that you would expect for input values of \(ti = 1\), \(yi = 1\), \(tf = 5\), \(n = 20\)?

**Solution:** The time step is given by \(h = (tf - ti)/n = (5 - 1)/20 = 1/5 = .2\). The initial time and data are given by \(t(1) = ti = 1\) and \(y(1) = yi = 1\). One then has

\[
t(2) = t(1) + h = 1 + .2 = 1.2,
\]

\[
z = t(1)^4 + y(1)^2 = 1 + 1 = 2,
\]

\[
y(2) = y(1) + (h/2) \left( z + t(2)^4 + (y(1) + h z)^2 \right) = 1 + .1(2 + (1.2)^4 + (1 + .2 \cdot 2)^2).
\]