Quiz 1 Solutions, Math 246, Professor David Levermore  
Tuesday, 8 September 2009

(1) [4] For each of the following ordinary differential equations, give its order and state whether it is linear or nonlinear.

(a) \( \frac{d^4w}{dz^4} + z^2 \frac{dw}{dz} + e^z w = z \);

\textbf{Solution:} fourth-order, linear.

(b) \( \frac{d^3y}{dx^3} - 2x \frac{dy}{dx} = \sin(y) \).

\textbf{Solution:} third-order, nonlinear.

(2) [2] Give the interval of definition for the solution of the initial-value problem

\( \frac{dx}{dt} + \tan(t) x = \frac{1}{t^2 - 4}, \quad x(3) = 7 \).

(You do not have to solve this equation to answer this question!)

\textbf{Solution:} This problem is linear in \( x \) and is already in normal form. The coefficient \( \tan(t) \) is continuous everywhere except where \( t = \frac{\pi}{2} + n\pi \) for some integer \( n \), while the forcing \( 1/(t^2 - 4) \) is continuous everywhere except at \( t = \pm 2 \). You can therefore read off that the interval of definition is \((2, \frac{3\pi}{2})\), the endpoints of which bracket the initial time 3 and are points where \( 1/(t^2 - 4) \) and \( \tan(t) \) respectively do not exist.

(3) [4] Solve the initial-value problem

\( z^2 \frac{dy}{dz} = y, \quad y(1) = e \).

\textbf{Solution:} This is a homogeneous linear equation. Its normal form is

\( \frac{dy}{dz} - \frac{1}{z^2} y = 0 \).

It has a general solution \( y = e^{-A(z)} c \), where \( A'(z) = -1/z^2 \). Let \( A(z) = 1/z \), so that \( y = e^{-1/z} c \). The initial condition then implies that \( e = e^{-1/1} c \), which yields \( c = e^2 \). The solution of the initial-value problem is therefore

\( y = e^{2 - \frac{1}{z}} \).

Notice that its interval of definition is \((0, \infty)\).

\textbf{Alternative Solution:} This is also a separable equation. Its separated differential form is

\( \frac{dy}{y} = \frac{dz}{z^2} \).

Its solutions are given implicitly by \( G(y) = F(z) + c \) where \( G'(y) = 1/y \) and \( F'(z) = 1/z^2 \). Let \( G(y) = \log(|y|) \) and \( F(z) = -1/z \), so that \( \log(|y|) = -1/z + c \). The initial condition then implies that \( \log(e) = -1/1 + c \), which yields \( c = 1 + \log(e) = 1 + 1 = 2 \).

Hence, \( \log(|y|) = 2 - \frac{1}{z} \). When you solve this for \( y \) you obtain \( y = \pm e^{2 - \frac{1}{z}} \). You then take the positive solution to match the initial condition.