1. [10] Let $X$ be a field. Use the field axioms to show that if $x \in X$ then $x0 = 0$.

**Remark:** The main point to keep in mind when doing problems like this is *to justify every step in your solution either by one or more of the axioms or by a previous step.*

**Solution.** Let $x \in X$. The additive identity axiom implies $0 = 0 + 0$. The distributive axiom then gives the equality

$$x0 = x(0 + 0) = x0 + x0.$$  

Then

$$0 = x0 + \left( - (x0) \right) \quad \text{(add. inv. axiom)}$$

$$= (x0 + x0) + \left( - (x0) \right) \quad \text{(above equality)}$$

$$= x0 + \left( x0 + \left( - (x0) \right) \right) \quad \text{(add. assoc. axiom)}$$

$$= x0 + 0 \quad \text{(add. inv. axiom)}$$

$$= x0 \quad \text{(add. indent. axiom)}.$$  

□

2. [10] Suppose that $a \in \mathbb{R}$ has the property that $a < 1/k$ for every $k \in \mathbb{Z}_+$. Prove $a \leq 0$.

**Solution.** Suppose $a \leq 0$ does not hold. Then by trichotomy $a > 0$. By the Archimedean Property there exists $n \in \mathbb{Z}_+$ such that $1 < na$. Then $1/n < a$, which contradicts the property that $a < 1/k$ for every $k \in \mathbb{Z}_+$. Therefore $a \leq 0$ holds. □

**Remark.** An alternative solution that uses more advanced machinery (and therefore is not as good) is the following. Because constant sequences converge while $1/k \to 0$ as $k \to \infty$, and because of the way limits preserve inequalities, one has

$$a = \lim_{k \to \infty} a \leq \lim_{k \to \infty} 1/k = 0. \quad \Box$$

The Archimedean Property lies behind the fact that $1/k \to 0$ as $k \to \infty$ in this alternative solution. □

3. [10] Write down a counterexample to each of the following assertions.

(a) A sequence $\{a_k\}_{k \in \mathbb{N}}$ in $\mathbb{R}$ is convergent if the sequence $\{a_k^2\}_{k \in \mathbb{N}}$ is convergent.

(b) A countable union of closed subsets of $\mathbb{R}$ is closed.

(c) Every convergent series in $\mathbb{R}$ is absolutely convergent.

**Solution (a).** Let $a_k = (-1)^k$ for every $k \in \mathbb{N}$. Then the sequence $\{a_k^2\}_{k \in \mathbb{N}}$ converges to 1 (because $a_k^2 = (-1)^{2k} = 1$), while the sequence $\{a_k\}_{k \in \mathbb{N}}$ diverges. □

**Solution (b).** Let $I_k = [2^{-k}, 2]$ for every $k \in \mathbb{N}$. Then each interval $I_k$ is closed while

$$\bigcup_{k \in \mathbb{N}} I_k = (0, 2]$$

is not closed. □
Solution (c). The series
\[ \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \]
is convergent,

by the Alternating Series Test, but is not absolutely convergent because the harmonic series,
\[ \sum_{k=1}^{\infty} \frac{1}{k}, \]
is divergent. □

4. [10] Consider the real sequence \( \{b_k\}_{k \in \mathbb{N}} \) given by
\[ b_k = (-1)^k \frac{2k + 4}{k + 1} \quad \text{for every } k \in \mathbb{N} = \{0, 1, 2, \ldots \}. \]

(a) Write down the first three terms of the subsequence \( \{b_{2k}\}_{k \in \mathbb{N}} \).
(b) Write down the first three terms of the subsequence \( \{b_{2k}\}_{k \in \mathbb{N}} \).
(c) Write down \( \lim \inf_{k \to \infty} b_k \) and \( \lim \sup_{k \to \infty} b_k \). (No proof is needed here.)

\textbf{Solution.} You are given that \( \mathbb{N} = \{0, 1, 2, \ldots \} \), as was done in class and in the notes (but in not the book). Then (a) the first three terms of the subsequence \( \{b_{2k}\}_{k \in \mathbb{N}} \) are
\[ b_0 = 4, \quad b_2 = \frac{8}{3}, \quad b_4 = \frac{12}{5}, \]
while (b) the first three terms of the subsequence \( \{b_{2k}\}_{k \in \mathbb{N}} \) are
\[ b_1 = -3, \quad b_2 = \frac{8}{3}, \quad b_4 = \frac{12}{5}. \]

Because \( b_{2k+1} < -2 \) while \( b_{2k} > 2 \), and because
\[ \lim_{k \to \infty} b_{2k+1} = - \lim_{k \to \infty} \frac{4k + 6}{2k + 2} = -2, \]
while
\[ \lim_{k \to \infty} b_{2k} = \lim_{k \to \infty} \frac{4k + 4}{2k + 1} = 2, \]
(c) one has that
\[ \lim \inf_{k \to \infty} b_k = -2, \quad \lim \sup_{k \to \infty} b_k = 2. \]

5. [10] Let \( \{a_k\}_{k \in \mathbb{N}} \) and \( \{b_k\}_{k \in \mathbb{N}} \) be bounded sequences in \( \mathbb{R} \).

(a) Prove that
\[ \lim \inf_{k \to \infty} a_k + \lim \inf_{k \to \infty} b_k \leq \lim \inf_{k \to \infty} (a_k + b_k). \]

(b) Write down an example for which equality does not hold above.

\textbf{Solution (a):} Let \( c_k = a_k + b_k \) for every \( k \in \mathbb{N} \). By the definition of \( \lim \inf \) we have
\[ \lim \inf_{k \to \infty} a_k = \lim_{k \to \infty} a_k, \quad \lim \inf_{k \to \infty} b_k = \lim_{k \to \infty} b_k, \quad \lim \inf_{k \to \infty} c_k = \lim_{k \to \infty} c_k, \]
where for every \( k \in \mathbb{N} \) we define
\[ a_k = \inf \{a_l : l \geq k\}, \quad b_k = \inf \{b_l : l \geq k\}, \quad c_k = \inf \{c_l : l \geq k\}. \]
Because the sequences \( \{a_k\}_{k \in \mathbb{N}}, \{b_k\}_{k \in \mathbb{N}}, \) and \( \{c_k\}_{k \in \mathbb{N}} \) are bounded below, for every \( k \in \mathbb{N} \) we have

\[-\infty < a_k, \quad -\infty < b_k, \quad -\infty < c_k.\]

Therefore \( \{a_k\}_{k \in \mathbb{N}}, \{b_k\}_{k \in \mathbb{N}}, \) and \( \{b_k\}_{k \in \mathbb{N}} \) are nondecreasing sequences in \( \mathbb{R} \). Moreover, they are bounded above because \( \{a_k\}_{k \in \mathbb{N}}, \{b_k\}_{k \in \mathbb{N}}, \) and \( \{c_k\}_{k \in \mathbb{N}} \) are bounded above. Hence, they converge by the Monotonic Sequence Theorem:

\[
\liminf_{k \to \infty} a_k = \lim_{k \to \infty} a_k, \quad \liminf_{k \to \infty} b_k = \lim_{k \to \infty} b_k, \quad \liminf_{k \to \infty} c_k = \lim_{k \to \infty} c_k. 
\]

Because for every \( k \in \mathbb{N} \) we have

\[
a_k + b_k \leq a_l + b_l = c_l \quad \text{for every} \quad l \geq k, \]

the sequences \( \{a_k\}_{k \in \mathbb{N}}, \{b_k\}_{k \in \mathbb{N}}, \) and \( \{c_k\}_{k \in \mathbb{N}} \) thereby also satisfy the inequality

\[
a_k + b_k \leq \inf \{c_l : l \geq k \} = c_k. 
\]

Then by the properties of limits

\[
\liminf_{k \to \infty} a_k + \liminf_{k \to \infty} b_k = \lim_{k \to \infty} a_k + \lim_{k \to \infty} b_k = \lim_{k \to \infty} (a_k + b_k) \\
\leq \lim_{k \to \infty} c_k = \liminf_{k \to \infty} c_k = \liminf_{k \to \infty} (a_k + b_k). 
\]

**Solution (b):** Let \( a_k = (-1)^k \) and \( b_k = (-1)^{k+1} \) for every \( k \in \mathbb{N} \). Clearly

\[
\liminf_{k \to \infty} a_k = \lim_{k \to \infty} a_{2k+1} = -1, \quad \liminf_{k \to \infty} b_k = \lim_{k \to \infty} b_{2k} = -1, 
\]

while (because \( a_k + b_k = 0 \) for every \( k \in \mathbb{N} \))

\[
\liminf_{k \to \infty} (a_k + b_k) = \lim_{k \to \infty} (a_k + b_k) = 0. 
\]

Therefore

\[
\liminf_{k \to \infty} a_k + \liminf_{k \to \infty} b_k = -2 < 0 = \liminf_{k \to \infty} (a_k + b_k). 
\]

6. [10] Let \( A \) and \( B \) be subsets of \( \mathbb{R} \).

(a) Prove that \( (A \cap B)^c \subset A^c \cap B^c \).

(b) Write down an example for which equality does not hold above.

**Solution (a):** Let \( x \in (A \cap B)^c \). By the definition of closure, there exists a sequence \( \{x_k\}_{k \in \mathbb{N}} \) contained in \( A \cap B \) such that \( x_k \to x \) as \( k \to \infty \). But the sequence \( \{x_k\}_{k \in \mathbb{N}} \) is therefore contained in both \( A \) and \( B \) while \( x_k \to x \) as \( k \to \infty \). By the definition of closure, it follows that \( x \in A^c \) and \( x \in B^c \), whereby \( x \in A^c \cap B^c \). □

**Solution (b):** A simple example is \( A = (0, 1) \) and \( B = (1, 2) \). Then \( (A \cap B)^c = \emptyset^c = \emptyset \) (because \( A \cap B = \emptyset \)), while \( A^c \cap B^c = [0, 1] \cap [1, 2] = \{1\} \) (because \( A^c = [0, 1] \) and \( B^c = [1, 2] \)). Hence, \( \emptyset = (A \cap B)^c \neq A^c \cap B^c = \{1\} \). □

**Remark.** A more dramatic example is \( A = \mathbb{Q} \) and \( B = \{\sqrt{2} + q : q \in \mathbb{Q}\} \). Then \( (A \cap B)^c = \emptyset^c = \emptyset \) (because \( A \cap B = \emptyset \)), while \( A^c \cap B^c = \mathbb{R} \cap \mathbb{R} = \mathbb{R} \) (because \( A^c = \mathbb{R} \) and \( B^c = \mathbb{R} \)). Hence, \( \emptyset = (A \cap B)^c \neq A^c \cap B^c = \mathbb{R} \). □
7. [20] Determine all \( a \in \mathbb{R} \) for which the following formal infinite series converge. Give your reasoning.

(a) \( \sum_{n=1}^{\infty} \frac{a^n}{n} \)

(b) \( \sum_{k=0}^{\infty} \left( \frac{k^2 + 1}{k^6 + 1} \right)^a \)

Solution (a). The series converges for \( a \in [-1, 1) \) and diverges otherwise.

The cases \( |a| < 1 \) and \( |a| > 1 \) are best handled by the Ratio Test. Let \( b_n = a^n / n \).

Because

\[
\lim_{n \to \infty} \frac{|b_{n+1}|}{|b_n|} = \lim_{n \to \infty} \frac{n}{n+1} |a| = |a|
\]

the Ratio Test shows that this series converges absolutely for \( |a| < 1 \) and diverges for \( |a| > 1 \). The Root Test would lead to the same conclusions.

The case \( a = -1 \) is best handled by the Alternating Series Test. Indeed, because the sequence

\[
\left\{ \frac{1}{n} \right\}_{n=1}^{\infty}
\]

is decreasing and positive, while \( \lim_{n \to \infty} \frac{1}{n} = 0 \),

the Alternating Series Test shows that

\[
\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}
\]

converges.

The case \( a = 1 \) reduces to the harmonic series, which diverges. Had you forgotten this fact, you could recover it with either the Cauchy \( 2^k \) Test or the Integral Test. □

Solution (b). The series converges for \( a \in (\frac{1}{4}, \infty) \) and diverges otherwise. Because

\[
\frac{k^2 + 1}{k^6 + 1} \sim \frac{1}{k^4} \quad \text{as} \quad k \to \infty,
\]

one sees that the original series should be compared with the \( p \)-series

\[
\sum_{k=1}^{\infty} \frac{1}{k^{4a}}.
\]

This is best handled by Limit Comparison Test. Indeed, because for every \( a \in \mathbb{R} \) one has

\[
\lim_{k \to \infty} \left( \frac{k^2 + 1}{k^6 + 1} \right)^a = \lim_{k \to \infty} \left( \frac{k^6 + k^4}{k^6 + 1} \right)^a = 1,
\]

the Limit Comparison Test then implies that

\[
\sum_{k=1}^{\infty} \left( \frac{k^2 + 1}{k^6 + 1} \right)^a \text{ converges} \iff \sum_{k=1}^{\infty} \frac{1}{k^{4a}} \text{ converges}.
\]

Because the \( p = 4a \) for the \( p \)-series, it converges for \( a \in (\frac{1}{4}, \infty) \) and diverges otherwise. The same is therefore true for the original series. □
8. [10] Let \( \{a_k\}_{k \in \mathbb{N}} \) be a real sequence and \( \{a_{n_k}\} \) be any subsequence of it. Show that

\[
\sum_{k=0}^{\infty} a_k \text{ converges absolutely } \implies \sum_{k=0}^{\infty} a_{n_k} \text{ converges absolutely}.
\]

**Solution.** For every \( m, n \in \mathbb{N} \) define the sequences \( \{p_m\} \) and \( \{q_n\} \) of partial sums

\[
p_m = \sum_{k=0}^{m} |a_{n_k}|, \quad q_n = \sum_{k=0}^{n} |a_k|.
\]

It is clear from these definitions that these sequences are nondecreasing and satisfy

\[
p_m = \sum_{k=0}^{m} |a_{n_k}| \leq \sum_{k=0}^{n_m} |a_k| = q_{n_m} \quad \text{for every } m \in \mathbb{N}.
\]

It then follows from the definition of absolute convergence, the Monotonic Sequence Theorem, and the above inequality that

\[
\sum_{k=0}^{\infty} a_k \text{ converges absolutely } \iff \{q_n\} \text{ converges}
\]

\[
\iff \{q_n\} \text{ is bounded above}
\]

\[
\implies \{p_m\} \text{ is bounded above}
\]

\[
\iff \{p_m\} \text{ converges}
\]

\[
\iff \sum_{k=0}^{\infty} a_{n_k} \text{ converges absolutely}.
\]

\[\square\]

9. [10] Let \( \{b_k\}_{k \in \mathbb{N}} \) be a sequence in \( \mathbb{R} \) and let \( A \) be a subset of \( \mathbb{R} \). Write the negations of the following assertions.

(a) “For some \( \epsilon > 0 \) one has \( |b_j - 3| \geq \epsilon \) frequently as \( j \to \infty \).”

(b) “Every sequence in \( A \) has a subsequence that converges to a limit in \( A \).”

**Solution (a).** “For every \( \epsilon > 0 \) one has \( |b_j - 3| < \epsilon \) eventually as \( j \to \infty \).” \[\square\]

**Solution (b).** “There is a sequence in \( A \) such that no subsequence of it converges to a limit in \( A \).” \[\square\]

Or better

“There is a sequence in \( A \) such that every subsequence of it either diverges or converges to a limit outside \( A \).” \[\square\]