Remark. This was the hardest final exam I have ever given to a Math 410 class, mostly due to its length. It was a challenge to an exceptional class. Many in the class rose to the challenge, with about two thirds earning an A or B. Every problem was done correctly by at least one student, although no student did them all correctly. I do not plan to give a Math 410 final exam that is this hard again. — D.L.

1. [10] Let \( \{x_n\}_{n \in \mathbb{N}} \) be a sequence in \( \mathbb{R} \). Give negations of each of the following assertions.

(a) For every \( \epsilon > 0 \) there exists an \( n_\epsilon \in \mathbb{N} \) such that
\[
m, n > n_\epsilon \implies |x_m - x_n| < \epsilon .
\]

Solution. There exists an \( \epsilon > 0 \) such that for every \( N \in \mathbb{N} \) there exists \( m, n \in \mathbb{N} \) such that
\[
m, n > N \quad \text{and} \quad |x_m - x_n| \geq \epsilon .
\]

(b) \( \lim_{n \to \infty} x_n = \infty \).

Solution. There are several acceptable answers. The shortest is
\[
\lim \inf_{n \to \infty} x_n < \infty .
\]

This could be expanded as
\[
\exists M > 0 \quad \text{such that} \quad x_n \leq M \quad \text{frequently as} \quad n \to \infty ,
\]
which could be expanded further as
\[
\exists M > 0 \quad \text{such that} \quad \forall m \in \mathbb{N} \quad \exists n > m \quad \text{such that} \quad x_n \leq M .
\]

You can also obtain the last two answers by first expressing \( \lim_{n \to \infty} x_n = \infty \) either as
\[
\forall M > 0 \quad x_n > M \quad \text{eventually as} \quad n \to \infty ,
\]
or as
\[
\forall M > 0 \quad \exists m \in \mathbb{N} \quad \text{such that} \quad \forall n > m \quad x_n > M ,
\]
and then simply negating.

2. [10] Let \( \{a_k\}_{k \in \mathbb{N}} \) be a nondecreasing sequence in \( \mathbb{R} \). Show that it converges if it has a converging subsequence.

Solution. Let \( \{a_{n_k}\}_{k \in \mathbb{N}} \) be a converging subsequence of \( \{a_k\}_{k \in \mathbb{N}} \). Because every subsequence of a nondecreasing sequence is also nondecreasing, the Monotonic Sequence Theorem states that the convergence of \( \{a_{n_k}\}_{k \in \mathbb{N}} \) implies that
\[
\lim_{k \to \infty} a_{n_k} = \sup \{a_{n_k} : k \in \mathbb{N}\} < \infty .
\]

Because for every \( k \in \mathbb{N} \) we have \( k \leq n_k \), the fact \( \{a_k\}_{k \in \mathbb{N}} \) is nondecreasing implies that
\[
a_k \leq a_{n_k} \leq \sup \{a_{n_k} : k \in \mathbb{N}\} < \infty .
\]

The nondecreasing sequence \( \{a_k\}_{k \in \mathbb{N}} \) is thereby bounded above, and therefore converges by the Monotonic Sequence Theorem. \( \square \)
3. [20] State whether each of the following statements is true or false. Give a proof when true and a counterexample when false.

(a) If the interval \((a, b)\) is bounded, \(f : (a, b) \to \mathbb{R}\) is differentiable, and \(f' : (a, b) \to \mathbb{R}\) is bounded over \((a, b)\) then the function \(f\) is bounded over \((a, b)\).

**Solution.** This statement is true. First notice that because \(f : (a, b) \to \mathbb{R}\) is differentiable the Mean-Value Theorem implies that for every \(x, y \in (a, b)\) there exists \(p \in (a, b)\) between \(x\) and \(y\) such that
\[
f(x) - f(y) = f'(p)(x - y).
\]
Because \(f' : (a, b) \to \mathbb{R}\) is bounded this implies that for every \(x, y \in (a, b)\) one has
\[
|f(x) - f(y)| = M|x - y|,
\]
where \(M = \sup\{|f'(x)| : x \in (a, b)\}\). In other words, \(f\) is Lipschitz continuous over \((a, b)\). This was a theorem from the notes that you could have just cited.
Finally, any function that is Lipschitz continuous over a bounded subset of \(\mathbb{R}\) is also bounded. Indeed, pick any \(c \in (a, b)\). Then for every \(x \in (a, b)\) one has the bound
\[
|f(x)| \leq |f(c)| + |f(x) - f(c)| \leq |f(c)| + M|x - c| \leq |f(c)| + M(b - a).
\]
More generally, any function that is uniformly continuous over a bounded subset of \(\mathbb{R}\) is also bounded.

(b) If \(\{f_n\}_{n=1}^\infty\) is a sequence of functions such that each \(f_n : [a, b] \to \mathbb{R}\) is differentiable over \([a, b]\), and \(f_n \to f\) uniformly over \([a, b]\) where \(f : [a, b] \to \mathbb{R}\) is differentiable over \([a, b]\), then
\[
\lim_{n \to \infty} f_n'(x) = f'(x) \quad \text{for every } x \in [a, b].
\]

**Solution.** This statement is false. There are many counterexamples. A simple one is
\[
f_n(x) = \frac{1}{n} e^{-nx} \to 0 = f(x) \quad \text{uniformly over } [0, 1] \text{ because } |f_n(x)| \leq \frac{1}{n},
\]
but
\[
f_n'(0) = -e^{-nx} \bigg|_{x=0} = -1 \neq 0 = f'(0).
\]
A more dramatic counterexample is
\[
f_n(x) = \frac{1}{2^n} \sin(2^nx) \to 0 = f(x) \quad \text{uniformly over } \mathbb{R} \text{ because } |f_n(x)| \leq \frac{1}{2^n},
\]
but if \(x = 2^{-k}m\pi\) for some \(k, m \in \mathbb{N}\) then
\[
f_n'(x) = \cos(2^{-k}m\pi) \to 1 \neq 0 = f'(x).
\]
Because the set of all points having the form \(2^{-k}m\pi\) for some \(k, m \in \mathbb{N}\) is dense in \(\mathbb{R}\), this example works when the functions \(f_n\) and \(f\) are restricted to any interval \([a, b] \subset \mathbb{R}\) with \(a < b\). □
4. [20] Let \( f : (a, b) \to \mathbb{R} \) be differentiable at a point \( c \in (a, b) \) with \( f'(c) > 0 \). Show that there exists a \( \delta > 0 \) such that

\[
\begin{align*}
    x \in (c - \delta, c) \subset (a, b) & \implies f(x) < f(c), \\
    x \in (c, c + \delta) \subset (a, b) & \implies f(c) < f(x),
\end{align*}
\]

**Remark.** It is very incorrect to assert that \( f \) is decreasing in an interval containing \( c \). You are being asked to prove the “Transversality Lemma” from the notes.

**Solution.** Because \( f \) is differentiable at \( c \), we have

\[
\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c).
\]

Because \( f'(c) > 0 \) the \( \epsilon, \delta \) characterization of this limit with \( \epsilon = f'(c) \) implies that there exists \( \delta > 0 \) such that \((c - \delta, c + \delta) \subset (a, b)\) and

\[
0 < |x - c| < \delta \implies \left| \frac{f(x) - f(c)}{x - c} - f'(c) \right| < f'(c) \iff 0 < \frac{f(x) - f(c)}{x - c} < 2f'(c).
\]

Hence,

\[
\begin{align*}
    x \in (c - \delta, c) & \implies f(x) - f(c) = \frac{f(x) - f(c)}{x - c}(x - c) < 0 \\
    & \implies f(x) < f(c), \\
    x \in (c, c + \delta) & \implies f(x) - f(c) = \frac{f(x) - f(c)}{x - c}(x - c) > 0 \\
    & \implies f(x) > f(c).
\end{align*}
\]

\( \square \)

5. [20] Let \( f(x) = \log(1 + x^2) \) for every \( x \in \mathbb{R} \). Show that

\[
f(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^{2k}
\]

for every \( x \in [-1, 1] \),

and that the series diverges for all other \( x \in \mathbb{R} \).

**Partial Solution.** It is easy to show that the series converges for every \( x \in [-1, 1] \) and diverges otherwise.

The convergence when \( |x| \leq 1 \) is best handled by the Alternating Series Test. Indeed, because the sequence

\[
\left\{ \frac{1}{k} x^{2k} \right\}_{k=1}^{\infty}
\]

is decreasing and positive,

and because

\[
\lim_{k \to \infty} \frac{1}{k} x^{2k} = 0,
\]

the Alternating Series Test shows that

\[
\sum_{k=1}^{\infty} \frac{(-1)^{k}}{k} x^{2k}
\]

converges.
The divergence when \(|x| > 1\) follows from the Divergence Test because in that case
\[
\lim_{k \to \infty} \frac{1}{k} x^{2k} = \infty \neq 0.
\]
However, this is not a complete solution to the problem because these arguments do not show that when the series converges, it converges to \(f(x)\).

**Solution.** Because \(x^2 \in [0, \infty)\) for every \(x \in \mathbb{R}\), the problem reduces to showing that
\[
\log(1 + y) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} y^k \quad \text{for every } y \in [0, 1],
\]
and that the series diverges for every \(y \in (1, \infty)\). The assertions will then follow upon setting \(y = x^2\) into these results.

Let \(g(y) = \log(1 + y)\). By direct computation you see that
\[
g'(y) = \frac{1}{1 + y}, \quad g''(y) = \frac{-1}{(1 + y)^2}, \quad g'''(y) = \frac{2}{(1 + y)^3}, \quad g''''(y) = \frac{-6}{(1 + y)^4}.
\]
This should suggest to you that for every \(k \in \mathbb{Z}_+\) one has
\[
g^{(k)}(y) = (-1)^{k+1} \frac{(k - 1)!}{(1 + y)^k},
\]
which is easily verified by induction. Because \(g(0) = 0\) while \(g^{(k)}(0) = (-1)^{k+1}(k - 1)!\) for every \(k \in \mathbb{Z}_+\), the Lagrange Remainder Theorem implies that for every \(y > 0\) there exists \(p \in (0, y)\) such that
\[
g(y) = \sum_{k=0}^{n} \frac{1}{k!} g^{(k)}(0) y^k + \frac{1}{(n + 1)!} g^{(n+1)}(p) y^{n+1}
\]
\[
= \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} y^k + \frac{(-1)^n}{n + 1} \frac{1}{(1 + p)^{n+1}} y^{n+1}.
\]
Because \(g(y) = \log(1 + y)\) while \(p \in (0, y)\), it follows that
\[
\left| \log(1 + y) - \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} y^k \right| = \frac{1}{n + 1} \frac{1}{(1 + p)^{n+1}} y^{n+1} < \frac{1}{n + 1} y^{n+1}.
\]
For every \(y \in [0, 1]\) we thereby obtain the uniform estimate
\[
\left| \log(1 + y) - \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} y^k \right| < \frac{1}{n + 1}.
\]
Hence,
\[
\log(1 + y) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} y^k \quad \text{uniformly over } y \in [0, 1].
\]
The fact that the series diverges for every \(y > 1\) follows from the Divergence Test because in that case
\[
\lim_{k \to \infty} \frac{1}{k} y^k = \infty \neq 0.
\]
The assertions follow by setting \(y = x^2\) into the above results. □
6. [20] Determine all \( a \in \mathbb{R} \) for which the following formal infinite series converge. Give your reasoning.

(a) \( \sum_{n=2}^{\infty} \frac{a^n}{3^n \log(n)} \)

**Solution.** The series converges for every \( a \in [-3, 3) \) and diverges otherwise.

The cases \( |a| < 3 \) and \( |a| > 3 \) are best handled by the Ratio Test. Let \( b_n = a^n/(3^n \log(n)) \). Because

\[
\lim_{n \to \infty} \frac{|b_{n+1}|}{|b_n|} = \lim_{n \to \infty} \frac{\log(n+1)}{\log(n)} \frac{|a|}{3} = \frac{|a|}{3},
\]

the Ratio Test then implies that this series converges absolutely for \( |a| < 3 \) and diverges for \( |a| > 3 \).

The case \( a = -3 \) is best handled by the Alternating Series Test. Indeed, because the sequence

\[
\left\{ \frac{1}{\log(n)} \right\}_{n=2}^{\infty}
\]

is decreasing and positive, and because

\[
\lim_{n \to \infty} \frac{1}{\log(n)} = 0,
\]

the Alternating Series Test shows that

\[
\sum_{n=2}^{\infty} (-1)^n \frac{1}{\log(n)}
\]

converges.

The case \( a = 3 \) is best handled by Limit Comparison Test, say with the harmonic series. Indeed, because

\[
\lim_{n \to \infty} \frac{\log(n)}{n} = 0,
\]

and because the harmonic series

\[
\sum_{n=2}^{\infty} \frac{1}{n}
\]

diverges,

the Limit Comparison Test shows that

\[
\sum_{n=2}^{\infty} \frac{1}{\log(n)}
\]

diverges.

Alternatively, one could treat this case with the Direct Comparison Test, the Integral Test, or the Cauchy \( 2^k \) Test. \( \square \)

(b) \( \sum_{k=1}^{\infty} \left( \frac{k}{k^4 + 1} \right)^a \)

**Solution.** The series converges for every \( a \in (\frac{1}{3}, \infty) \) and diverges otherwise.
This is best handled by Limit Comparison Test. Because
\[ \frac{k}{k^4 + 1} \sim \frac{1}{k^3} \quad \text{as} \quad k \to \infty, \]
one sees that the original series should be compared with the p-series
\[ \sum_{k=1}^{\infty} \frac{1}{k^{3a}}. \]
Indeed, because for every \( a \in \mathbb{R} \) one has
\[ \lim_{k \to \infty} \left( \frac{k}{k^4 + 1} \right)^a = \lim_{k \to \infty} \left( \frac{k^4}{k^4 + 1} \right)^a = 1, \]
the Limit Comparison Test then implies that
\[ \sum_{k=1}^{\infty} \left( \frac{k}{k^4 + 1} \right)^a \text{ converges } \iff \sum_{k=1}^{\infty} \frac{1}{k^{3a}} \text{ converges}. \]
Because \( p = 3a \) for the p-series, that series converges for \( a \in (\frac{1}{3}, \infty) \) and diverges otherwise. The same is therefore true for the original series. \( \square \)

7. [20] Let \( f : [a, b] \to \mathbb{R} \) and \( g : [a, b] \to \mathbb{R} \) be Riemann integrable over \([a, b]\). Show that \( f + g \) is Riemann integrable over \([a, b]\).

**Solution.** The shortest route to a proof uses the Lebesgue Theorem. Let \( D_f, D_g, \) and \( D_{f+g} \) denote the points in \([a, b]\) at which \( f, g, \) and \( f + g \) respectively are discontinuous. It is clear that \( D_{f+g} \subset D_f \cup D_g \) because \( f + g \) is continuous at every point where both \( f \) and \( g \) are continuous. Because \( f \) and \( g \) are Riemann integrable over \([a, b]\), one direction of the Lebesgue Theorem implies that \( D_f \) and \( D_g \) have measure zero. Because the union of two measure zero sets also has measure zero, and because any subset of a measure zero set also has measure zero, it follows that \( D_{f+g} \subset D_f \cup D_g \) has measure zero. The other direction of the Lebesgue Theorem then implies that \( f + g \) is Riemann integrable over \([a, b]\). \( \square \)

**Alternative Solution.** Let \( \epsilon > 0 \). Because \( f \) and \( g \) are Riemann integrable over \([a,b]\), the Darboux Theorem implies that there exist partitions \( P^I_\epsilon \) and \( P^g_\epsilon \) of \([a,b]\) such that
\[ 0 \leq U(f, P^I_\epsilon) - L(f, P^I_\epsilon) < \frac{\epsilon}{2}, \quad 0 \leq U(g, P^g_\epsilon) - L(f, P^g_\epsilon) < \frac{\epsilon}{2}. \]
Set \( P_\epsilon = P^I_\epsilon \vee P^g_\epsilon \). Then
\[ U(f + g, P_\epsilon) \leq U(f, P_\epsilon) + U(g, P_\epsilon) \leq U(f, P^I_\epsilon) + U(g, P^g_\epsilon), \]
\[ L(f + g, P_\epsilon) \geq L(f, P_\epsilon) + L(g, P_\epsilon) \geq L(f, P^I_\epsilon) + L(g, P^g_\epsilon). \]
Upon combining the above inequalities you find that
\[ 0 \leq U(f + g, P_\epsilon) - L(f + g, P_\epsilon) \]
\[ \leq U(f, P^I_\epsilon) - L(f, P^I_\epsilon) + U(g, P^g_\epsilon) - L(g, P^g_\epsilon) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \]
Because such a \( P_\epsilon \) can be found for every \( \epsilon > 0 \), the Darboux Theorem implies that \( f + g \) is Riemann integrable. \( \square \)
Remark. The second solution gives a lot more. It is just a few steps away from showing that the integral of \( f + g \) is the sum of the integrals of \( f \) and \( g \).

8. [20] A function \( f : [a, b] \to \mathbb{R} \) is said to be Hölder continuous of order \( \alpha \in (0, 1] \) if there exists a \( C \in \mathbb{R}_+ \) such that for every \( x, y \in [a, b] \) one has

\[
|f(x) - f(y)| < C |x - y|^\alpha.
\]

Let \( f : [a, b] \to \mathbb{R} \) be Hölder continuous of order \( \alpha \in (0, 1] \).

(a) Show that \( f \) is uniformly continuous over \([a, b] \).

Solution. Let \( \epsilon > 0 \). Set \( \delta = (\epsilon/C)^{1/\alpha} \). Then for every \( x, y \in [a, b] \) we have

\[
|x - y| < \delta \implies |f(x) - f(y)| < C |x - y|^\alpha < C \delta^\alpha = \epsilon.
\]

Hence, \( f \) is uniformly continuous over \([a, b] \). \( \square \)

(b) Show that for every partition \( P \) of \([a, b] \) one has

\[
0 \leq U(f, P) - L(f, P) < |P|^\alpha C (b - a).
\]

Solution. Let \( P = [x_0, x_1, \ldots, x_n] \) be any partition of \([a, b] \). Then

\[
0 \leq U(f, P) - L(f, P) = \sum_{k=1}^{n} (\overline{m}_k - \underline{m}_k)(x_k - x_{k-1}),
\]

where

\[
\overline{m}_k = \sup \{ f(x) : x \in [x_{k-1}, x_k] \}, \quad \underline{m}_k = \inf \{ f(x) : x \in [x_{k-1}, x_k] \}.
\]

Because \( f \) is continuous over each \([x_{k-1}, x_k] \), by the Extreme-Value Theorem there exists points \( \overline{x}_k, \underline{x}_k \in [x_{k-1}, x_k] \) such that \( \overline{m}_k = f(\overline{x}_k) \) and \( \underline{m}_k = f(\underline{x}_k) \). The Hölder continuity of \( f \) gives

\[
0 \leq U(f, P) - L(f, P) = \sum_{k=1}^{n} (f(\overline{x}_k) - f(\underline{x}_k))(x_k - x_{k-1})
\]

\[
\leq C \sum_{k=1}^{n} |\overline{x}_k - \underline{x}_k|^\alpha (x_k - x_{k-1}).
\]

Because \( \overline{x}_k, \underline{x}_k \in [x_{k-1}, x_k] \) you have

\[
|\overline{x}_k - \underline{x}_k| \leq x_k - x_{k-1} \leq \max \{ x_m - x_{m-1} : m = 1, \ldots, n \} \equiv |P|,
\]

whereby

\[
0 \leq U(f, P) - L(f, P) \leq C \sum_{k=1}^{n} |P|^\alpha (x_k - x_{k-1})
\]

\[
= C |P|^\alpha \sum_{k=1}^{n} (x_k - x_{k-1}) = C |P|^\alpha (b - a).
\]

\( \square \)
9. Let \( \{g_n\}_{n \in \mathbb{N}} \) be a sequence of real-valued functions over \( D \subset \mathbb{R} \), and \( \{M_n\}_{n \in \mathbb{N}} \) be a sequence of real numbers such that \( |g_n(x)| \leq M_n \) for every \( x \in D \) and \( n \in \mathbb{N} \). Show that

\[
\sum_{n=0}^{\infty} M_n < \infty \implies \sum_{n=0}^{\infty} g_n(x) \text{ converges uniformly over } D.
\]

Remark: You are being asked to prove the Weierstrass M-Test.

**Solution.** The Absolute Comparison Test states that

\[
\sum_{n=0}^{\infty} M_n < \infty \implies \sum_{n=0}^{\infty} g_n(x) \text{ converges absolutely for every } x \in D.
\]

You must show that this pointwise convergence is uniform. Equivalently, if you introduce

\[
f_n(x) = \sum_{k=0}^{n} g_k(x), \quad f(x) = \sum_{k=0}^{\infty} g_k(x),
\]

then you must show that \( f_n \to f \) uniformly over \( D \).

Let \( \epsilon > 0 \). Because \( \sum_{k=0}^{\infty} M_k < \infty \) there exists \( n_\epsilon \in \mathbb{N} \) such that

\[
n > n_\epsilon \implies \sum_{k=n+1}^{\infty} M_k < \epsilon.
\]

Then for every \( n > n_\epsilon \) and every \( x \in D \) one has

\[
|f_n(x) - f(x)| = \left| \sum_{k=1}^{n} g_k(x) - \sum_{k=1}^{\infty} g_k(x) \right| = \left| \sum_{k=n+1}^{\infty} g_k(x) \right| \\
\leq \sum_{k=n+1}^{\infty} |g_k(x)| \leq \sum_{k=n+1}^{\infty} M_k < \epsilon.
\]

Hence, \( f_n \to f \) uniformly over \( D \).

10. For each \( n \in \mathbb{Z}_+ \) define \( f_n : [0, 1] \to \mathbb{R} \) by \( f_n(x) = n^2 xe^{-nx} \).

(a) Sketch the graph of a typical \( f_n \) over \([0, 1]\).

**Solution.** Because \( f'_n(x) = n^2(1-nx)e^{-nx} \) is positive over \([0, \frac{1}{n}]\) and negative over \((\frac{1}{n}, 1]\) for any \( n > 1 \) your sketch should show that:

- the value of \( f_n(x) \) increases over \([0, \frac{1}{n}]\) from \( f_n(0) = 0 \) at \( x = 0 \) to a maximum of \( f_n(\frac{1}{n}) = \frac{n}{e} \) at \( x = \frac{1}{n} \);
- the value of \( f_n(x) \) decreases over \((\frac{1}{n}, 1]\) from its maximum of \( f_n(\frac{1}{n}) = \frac{n}{e} \) at \( x = \frac{1}{n} \) to \( f_n(1) = n^2e^{-n} \) at \( x = 1 \).

This shows that as \( n \) increases the maximum value of \( f_n \) increases as \( \frac{n}{e} \) while its location moves closer to \( x = 0 \). This understanding is helpful for the rest of the problem.
(b) Show that $f_n \to 0$ pointwise over $[0, 1]$.

**Solution.** Because $f_n(0) = 0$ for every $n \in \mathbb{N}$, the convergence of $\{f_n(x)\}$ when $x = 0$ is obvious. Now consider the sequence $\{f_n(x)\}$ for $x \in (0, 1]$. By l’Hôpital applied twice to the $\frac{\infty}{\infty}$ indeterminant form we see that

$$
\lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \frac{n^2 x}{e^{nx}} = \lim_{n \to \infty} \frac{2nx}{xe^{nx}} = \lim_{n \to \infty} \frac{2x}{x^2e^{nx}} = 0.
$$

Notice that the above derivatives are taken with respect to the variable $n$, which we consider extended to all $\mathbb{R}$ when applying l’Hôpital. Alternatively, rather than applying l’Hôpital you could argue that the limit vanishes because $e^{nx} \to \infty$ faster than $n^2x \to \infty$ when $x > 0$. □

(c) Show that the limit in part (b) is not uniform over $[0, 1]$.

**Solution.** You must show that there exists $\epsilon > 0$ such that for every $m \in \mathbb{N}$ there exists $n > m$ and $x \in [0, 1]$ such that $f_n(x) \geq \epsilon$. This is easy to do. In fact, for any $\epsilon > 0$ one has for every $n > \epsilon e$ that $f_n\left(\frac{1}{n}\right) = \frac{\epsilon}{e} > \epsilon$. □

(d) For every $\delta > 0$ show that $f_n \to 0$ uniformly over $[\delta, 1]$.

**Solution.** Let $\epsilon > 0$. You must show there exists $n_\epsilon \in \mathbb{N}$ such that for every $n > n_\epsilon$ and $x \in [\delta, 1]$ one has $0 < f_n(x) < \epsilon$. Here are two approaches to doing this.

**First Approach.** First notice that for every $x \in [\delta, 1]$ and $n > 0$ one has the inequalities

$$
0 < f_n(x) = n^2 xe^{-nx} < x^2 e^{-n\delta}.
$$

Either by noticing that $x^2 e^{-n\delta} = f_n(\delta)/\delta$ and using the pointwise convergence of assertion (b), or by arguing as in the proof of assertion (b) one sees that

$$
\lim_{n \to \infty} x^2 e^{-n\delta} = 0.
$$

Hence, there exists $n_\epsilon \in \mathbb{N}$ such that for every $n > n_\epsilon$ one has $x^2 e^{-n\delta} < \epsilon$. It follows that for every $n > n_\epsilon$ and $x \in [\delta, 1]$ one has

$$
0 < f_n(x) \leq x^2 e^{-n\delta} < \epsilon.
$$

Therefore $f_n \to 0$ uniformly over $[\delta, 1]$. □

**Second Approach.** First notice from part (a) that if $\frac{1}{n} < \delta$ then $f_n$ is decreasing over $[\delta, 1]$. Hence, for every $n > 1/\delta$ and $x \in [\delta, 1]$ one has

$$
0 < f_n(x) \leq f_n(\delta).
$$

By the pointwise convergence of assertion (b) applied to the point $x = \delta$ there exists $n_\epsilon \in \mathbb{N}$ such that $n_\epsilon > 1/\delta$ and that for every $n > n_\epsilon$ one has $f_n(\delta) < \epsilon$. It follows that for every $n > n_\epsilon$ and $x \in [\delta, 1]$ one has

$$
0 < f_n(x) \leq f_n(\delta) < \epsilon.
$$

Therefore $f_n \to 0$ uniformly over $[\delta, 1]$. □
(e) Show that
\[ \lim_{n \to \infty} \int_0^1 f_n = 1. \]

**Solution.** One integration by parts \((u = nx, v = -e^{-nx})\) yields
\[ \int_0^1 f_n = \int_0^1 n^2 x e^{-nx} \, dx = -n x e^{-nx} \bigg|_0^1 + \int_0^1 e^{-nx} n \, dx = -ne^{-n} - e^{-nx} \bigg|_0^1 = -ne^{-n} - e^{-n} + 1 = 1 - \frac{n + 1}{e^n}. \]

By l'Hôpital applied to the \(\frac{\infty}{\infty}\) indeterminant form we see that
\[ \lim_{n \to \infty} \frac{n + 1}{e^n} = \lim_{n \to \infty} \frac{1}{e^n} = 0. \]

Hence,
\[ \lim_{n \to \infty} \int_0^1 f_n = 1 - \lim_{n \to \infty} \frac{n + 1}{e^n} = 1. \]

(f) Let \(g : [0, 1] \to \mathbb{R}\) be continuous. Show that
\[ \lim_{n \to \infty} \int_0^1 f_n g = g(0). \]

**Solution.** Assertion (e) implies that assertion (f) is equivalent to
\[ \lim_{n \to \infty} \int_0^1 f_n(x) (g(x) - g(0)) \, dx = 0. \]

But this will follow once we show that for every \(\epsilon > 0\)
\[ \limsup_{n \to \infty} \int_0^1 f_n(x) \left| g(x) - g(0) \right| \, dx \leq \epsilon. \]

Let \(\epsilon > 0\). Because \(g\) is continuous at 0, there exists \(\delta > 0\) such that
\[ x \in [0, \delta) \implies \left| g(x) - g(0) \right| < \epsilon. \]

Because \(g\) is continuous over \([0, 1]\), by the Extreme-Value Theorem it is bounded over \([0, 1]\). Let \(M = \sup\{|g(x)| : x \in [0, 1]\}\). Then for every \(n \in \mathbb{N}\)
\[ \int_0^1 f_n(x) \left| g(x) - g(0) \right| \, dx = \int_0^\delta f_n(x) \left| g(x) - g(0) \right| \, dx + \int_\delta^1 f_n(x) \left| g(x) - g(0) \right| \, dx \leq \int_0^\delta f_n(x) \epsilon \, dx + \int_\delta^1 f_n(x) 2M \, dx \leq \epsilon \int_0^1 f_n(x) \, dx + 2M \int_\delta^1 f_n(x) \, dx. \]

Assertion (e) and the uniform convergence of assertion (d) imply that
\[ \lim_{n \to \infty} \int_0^1 f_n(x) \, dx = 1, \quad \lim_{n \to \infty} \int_\delta^1 f_n(x) \, dx = 0. \]
whereby the previous inequality implies that
\[
\limsup_{n \to \infty} \int_0^1 f_n(x) |g(x) - g(0)| \, dx \leq \epsilon \lim_{n \to \infty} \int_0^1 f_n(x) \, dx + 2M \lim_{n \to \infty} \int_\delta^1 f_n(x) \, dx = \epsilon.
\]
But as argued above, because this holds for every \( \epsilon > 0 \), assertion (f) follows. \( \square \)