1. Prove the divergence assertion of Proposition 3.11 in the notes.

2. Prove Proposition 3.12 in the notes.

3. Consider the set
   \[ \left\{ x \in \mathbb{R} : \sum_{n=0}^{\infty} \frac{(4n)! (2n)!}{n! (3n)!} x^n \text{ converges} \right\} . \]
   Use the ratio test to prove that this set is an interval and find its endpoints.

4. Determine all \( x, p \in \mathbb{R} \) for which the Fourier \( p \)-series
   \[ \sum_{k=1}^{\infty} \sin(kx) \frac{1}{k^p} \text{ converges} . \]

5. Let \( X \) be a field. Use the field axioms to show that if \( x, y \in X \) then \( (x^{-1} y)^{-1} = y^{-1} x \).

6. Let \( X \) be a field. Use the field axioms to show that if \( x, y \in X \) then \( (-x)(-y) = xy \).

7. Let \( A \) and \( B \) be closed subsets of \( \mathbb{R} \). Show that \( A \cap B \) and \( A \cup B \) are closed.

8. Consider the real sequence \( \{b_k\}_{k \in \mathbb{N}} \) given by
   \[ b_k = (-1)^k \left( 3 + \frac{1}{(k+1)^2} \right) \text{ for every } k \in \mathbb{N}, \]
   where \( \mathbb{N} = \{0, 1, 2, \cdots \} \).
   (a) Give the first three terms of the subsequence \( \{b_{3k}\}_{k \in \mathbb{N}} \).
   (b) Give the first three terms of the subsequence \( \{b_{2k-1}\}_{k \in \mathbb{N}} \).
   (c) Compute \( \limsup_{k \to \infty} b_k \) and \( \liminf_{k \to \infty} b_k \). Justify your answers.

9. Determine all the values of \( a \in \mathbb{R} \) for which
   \[ \sum_{n=2}^{\infty} \frac{1}{\log(n)} a^n \text{ converges} . \]

10. Determine all the values of \( a \in \mathbb{R} \) for which
    \[ \sum_{k=0}^{\infty} \left( \frac{2k+3}{k^4+1} \right)^a \text{ converges} . \]

11. Determine all the values of \( a \in \mathbb{R} \) for which
    \[ \sum_{m=1}^{\infty} \frac{1}{m^2} (2 + (-1)^m)^m a^m \text{ converges} . \]

12. Let \( \{b_k\}_{k \in \mathbb{N}} \) be a sequence in \( \mathbb{R} \) and let \( A \) be a subset of \( \mathbb{R} \). Write the negations of the following assertions.
    (a) “For every \( m \in \mathbb{R} \) one has \( b_j > m \) frequently as \( j \to \infty \).”
    (b) “Every sequence in \( A \) has a subsequence that converges to a limit in \( A \).”