Solutions to Second In-Class Exam: Math 401
Section 0201, Professor Levermore
Friday, 16 April 2010

No notes, books, or electrontics. You must show your reasoning for full credit. Good luck!

1. [20] Consider the polynomials

$$p_1(x) = 1 - x^2\,, \qquad p_2(x) = x(1-x)\,, \qquad p_3(x) = x(1+x)\,.$$

(a) Show that $\{p_1(x), p_2(x), p_3(x)\}$ is a basis for $\mathcal{P}^{(2)}$.
(b) Express $q(x) = 1 + 3x$ in terms of this basis.

Solution (a). The set $\{p_1(x), p_2(x), p_3(x)\}$ is related to the standard basis $\{1, x, x^2\}$ of $\mathcal{P}^{(2)}$ by

$$\begin{pmatrix} p_1(x) & p_2(x) & p_3(x) \end{pmatrix} = \begin{pmatrix} 1 & x & x^2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ -1 & -1 & 1 \end{pmatrix}.$$

Then $\{p_1(x), p_2(x), p_3(x)\}$ will be a basis for $\mathcal{P}^{(2)}$ if and only if the matrix above is invertible. There are many ways we can show this. For example, we can show that

$$\det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ -1 & -1 & 1 \end{pmatrix} = 1 - (-1) = 2\,,$$

or we can use row reduction to show that

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ -1 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

The invertiblity of the matrix then implies that $\{p_1(x), p_2(x), p_3(x)\}$ is a basis for $\mathcal{P}^{(2)}$.

Alternative Solution (a). It is clear that $\{p_1(x), p_2(x), p_3(x)\} \subset \mathcal{P}^{(2)}$. Because the dimension of $\mathcal{P}^{(2)}$ is three, we only have to show that the set $\{p_1(x), p_2(x), p_3(x)\}$ is linearly independent. We do this by showing that

$$c_1 p_1(x) + c_2 p_2(x) + c_3 p_3(x) = 0 \qquad \Longrightarrow \qquad c_1 = c_2 = c_3 = 0\,.$$

This can be done several ways. For example, upon evaluating the linear combination at $x = 0$, $x = -1$, and $x = 1$ we obtain

$$\begin{aligned} 0 &= c_1 p_1(0) + c_2 p_2(0) + c_3 p_3(0) = c_1\,, \\ 0 &= c_1 p_1(-1) + c_2 p_2(-1) + c_3 p_3(-1) = -2c_2\,, \\ 0 &= c_1 p_1(1) + c_2 p_2(1) + c_3 p_3(1) = 2c_3\,, \end{aligned}$$

which implies that $c_1 = c_2 = c_3 = 0$. Alternatively, if we express the linear combination in terms of the standard basis $\{1, x, x^2\}$ of $\mathcal{P}^{(2)}$ it becomes

$$0 = c_1(1 - x^2) + c_2(x - x^2) + c_3(x + x^2) = c_1 1 + (c_2 + c_3)x + (-c_1 - c_2 + c_3)x^2\,.$$

This leads to the system

$$c_1 = 0\,, \qquad c_2 + c_3 = 0\,, \qquad c_1 + c_2 - c_3 = 0\,,$$

which can be solved to show that $c_1 = c_2 = c_3 = 0$.

Solution (b). By setting

$$\begin{aligned} 1 + 3x = q(x) &= c_1 p_1(x) + c_2 p_2(x) + c_3 p_3(x) \\ &= c_1(1 - x^2) + c_2(x - x^2) + c_3(x + x^2) \\ &= c_1 1 + (c_2 + c_3)x + (-c_1 - c_2 + c_3)x^2 \,, \end{aligned}$$

we are led to the system

$$c_1 = 1 \,, \qquad c_2 + c_3 = 3 \,, \qquad c_1 + c_2 - c_3 = 0 \,,$$

which can be solved to find that $c_1 = c_2 = 1$ and $c_3 = 2$. We thereby obtain

$$q(x) = p_1(x) + p_2(x) + 2p_3(x) \,.$$

2. [15] Consider the directed graph

$$\begin{array}{ccc} 1 & \rightarrow & 2 \\ \uparrow & \nearrow & \downarrow \\ 4 & \leftarrow & 3 \end{array}$$

(a) Label the edges and give the corresponding incidence matrix A for the graph.
(b) Give a basis for the $\ker(A^T)$.

Solution (a). If we label the five edges a, b, c, d, and e as

$$\begin{array}{ccccc} & & a & & \\ & 1 & \longrightarrow & 2 & \\ d & \uparrow & e \nearrow & \downarrow & b \\ & 4 & \longleftarrow & 3 & \\ & & c & & \end{array}$$

then the corresponding incidence matrix A is

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \end{pmatrix} .$$

Here we used the convention that the vertex at the “base” of the arrow along an edge is assigned 1 while the vertex at the “point” is assigned -1. Had we flipped this convention then A would be the negative of the matrix given above.

Solution (b). Because A^T is 5×4 while $\text{rank}(A^T) = \text{rank}(A) = 4 - 1 = 3$, we know that $\ker(A^T)$ is two dimensional. By row reduction we see that

$$A^T = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ -1 & 1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & -1 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} .$$

We see there are two free parameters in the general solution, which is what we expect because $\ker(A^T)$ is two dimensional. We can obtain a basis for $\ker(A^T)$ by setting each parameter to 1 while setting the other to 0. This gives the basis

$$\begin{pmatrix}1\\1\\1\\1\\0\end{pmatrix}, \qquad \begin{pmatrix}0\\1\\1\\0\\1\end{pmatrix}.$$

The first of these corresponds to the cycle $abcd$ ($1 \to 2 \to 3 \to 4 \to 1$), while the second corresponds to the cycle bce ($2 \to 3 \to 4 \to 2$).

3. [20] Define the linear mapping $L(X) = AX$ for every 2×2 matrix X where $A = \begin{pmatrix}a & b\\ b & a\end{pmatrix}$. A basis for 2×2 matrices is given by

$$E_1 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \qquad E_2 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \qquad E_3 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \qquad E_4 = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}.$$

Give the matrix representative of L with respect to this basis.

Solution. Given any finite dimensional linear space $\mathcal{U}$ and linear mapping $L : \mathcal{U} \to \mathcal{U}$, the matrix representative of L with respect to a basis $\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_m\}$ of $\mathcal{U}$ is the $m\times m$ matrix R such that

$$\begin{pmatrix}L(\mathbf{u}_1) & L(\mathbf{u}_2) & \cdots & L(\mathbf{u}_m)\end{pmatrix} = \begin{pmatrix}\mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_m\end{pmatrix} R\,.$$

For the given basis $\{E_1, E_2, E_3, E_4\}$ of 2×2 matrices, direct calculations show that

$$\begin{aligned}
L(E_1) = AE_1 &= \begin{pmatrix}a & b\\ b & a\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}a & b\\ b & a\end{pmatrix} = aE_1 + bE_3\,,\\
L(E_2) = AE_2 &= \begin{pmatrix}a & b\\ b & a\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \begin{pmatrix}a & -b\\ b & -a\end{pmatrix} = aE_2 + bE_4\,,\\
L(E_3) = AE_3 &= \begin{pmatrix}a & b\\ b & a\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}b & a\\ a & b\end{pmatrix} = bE_1 + aE_3\,,\\
L(E_4) = AE_4 &= \begin{pmatrix}a & b\\ b & a\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}b & -a\\ a & -b\end{pmatrix} = aE_2 + bE_4\,.
\end{aligned}$$

Because these calculations show that

$$\begin{pmatrix}L(E_1) & L(E_2) & L(E_3) & L(E_4)\end{pmatrix} = \begin{pmatrix}E_1 & E_2 & E_3 & E_4\end{pmatrix}\begin{pmatrix}a & 0 & b & 0\\ 0 & a & 0 & b\\ b & 0 & a & 0\\ 0 & b & 0 & a\end{pmatrix},$$

the matrix representative of L with respect to the basis $\{E_1, E_2, E_3, E_4\}$ is

$$\begin{pmatrix}a & 0 & b & 0\\ 0 & a & 0 & b\\ b & 0 & a & 0\\ 0 & b & 0 & a\end{pmatrix}.$$

4. [15] Construct monic polynomials p_0, p_1, p_2, and p_3 of degrees 0, 1, 2, and 3 respectively that are orthogonal with respect to the inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(x)\, g(x)\, x^4 \mathrm{d}x\,.$$

Solution. By applying the Gram-Schmidt procedure to $\{1, x, x^2, x^3\}$ you obtain the monic orthogonal polynomials

$$\begin{aligned}
p_0(x) &= 1\,,\\
p_1(x) &= x - \frac{\langle 1, x\rangle}{\langle 1, 1\rangle}\, 1\,,\\
p_2(x) &= x^2 - \frac{\langle 1, x^2\rangle}{\langle 1, 1\rangle}\, 1 - \frac{\langle p_1, x^2\rangle}{\langle p_1, p_1\rangle}\, p_1(x)\,,\\
p_3(x) &= x^3 - \frac{\langle 1, x^3\rangle}{\langle 1, 1\rangle}\, 1 - \frac{\langle p_1, x^3\rangle}{\langle p_1, p_1\rangle}\, p_1(x) - \frac{\langle p_2, x^3\rangle}{\langle p_2, p_2\rangle}\, p_2(x)\,.
\end{aligned}$$

Because their integrands each have odd symmetry over $[-1, 1]$, you see that

$$\langle 1, x\rangle = \langle x, x^2\rangle = \langle 1, x^3\rangle = \langle x^2, x^3\rangle = 0\,,$$

while direct calculations show that

$$\begin{aligned}
\langle 1, 1\rangle &= \int_{-1}^{1} x^4 \mathrm{d}x = \tfrac{1}{5}x^5\Big|_{-1}^{1} = \tfrac{2}{5}\,,\\
\langle 1, x^2\rangle = \langle x, x\rangle &= \int_{-1}^{1} x^6 \mathrm{d}x = \tfrac{1}{7}x^7\Big|_{-1}^{1} = \tfrac{2}{7}\,,\\
\langle x, x^3\rangle = \langle x^2, x^2\rangle &= \int_{-1}^{1} x^8 \mathrm{d}x = \tfrac{1}{9}x^9\Big|_{-1}^{1} = \tfrac{2}{9}\,.
\end{aligned}$$

The Gram-Schmidt procedure therefore yields

$$\begin{aligned}
p_0(x) &= 1\,,\\
p_1(x) &= x - 0\cdot 1 = x\,,\\
p_2(x) &= x^2 - \tfrac{5}{7}\cdot 1 - 0\cdot x = x^2 - \tfrac{5}{7}\,,\\
p_3(x) &= x^3 - 0\cdot 1 - \tfrac{7}{9}\cdot x - 0\cdot (x^2 - \tfrac{5}{7}) = x^3 - \tfrac{7}{9}x\,,
\end{aligned}$$

where in the last step we used the fact that $\langle p_2, x^3\rangle = \langle x^2, x^3\rangle - \frac{5}{7}\langle 1, x^3\rangle = 0$.

5. [15] Find all real values of c for which the matrix

$$A = \begin{pmatrix} 1 & -1 & 0\\ -1 & c & -1\\ 0 & -1 & 1\end{pmatrix} \qquad \text{is Hermitian positive}\,.$$

Be sure to give your reasoning!

Solution. Because c is real, the matrix A is Hermitian. It will therefore be Hermitian positive if and only if the determinants of its principle minors are all positive. These determinants are

$$\det(1) = 1\,, \qquad \det\begin{pmatrix} 1 & -1 \\ -1 & c \end{pmatrix} = c - 1\,, \qquad \det\begin{pmatrix} 1 & -1 & 0 \\ -1 & c & -1 \\ 0 & -1 & 1 \end{pmatrix} = c - 2\,.$$

It is clear that all of these determinants are positive if and only if $c > 2$. Therefore A is Hermitian positive if and only if $c > 2$.

Alternative Solution. Because c is real, the matrix A is Hermitian. It will therefore be Hermitian positive if and only if all of its pivots are positive. By row reduction we obtain

$$A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & c & -1 \\ 0 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 \\ 0 & c-1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 \\ 0 & c-1 & -1 \\ 0 & 0 & 1 - \frac{1}{c-1} \end{pmatrix},$$

where we must assume that $c \neq 1$. The pivots are given by

$$1\,, \qquad c - 1\,, \qquad 1 - \frac{1}{c-1} = \frac{c-2}{c-1}\,.$$

It is clear that all of these pivots are positive if and only if $c > 2$. Therefore A is Hermitian positive if and only if $c > 2$.

Remark. The two approaches to the solution given above are related by the general fact that the n^{th} pivot of A for $n > 1$ is given by $\det(A_n)/\det(A_{n-1})$, where A_n denotes the n^{th} principle minor matrix of A, while the first pivot of A is $a_{11} = \det(A_1)$. So showing that the pivots all are positive is equivalent to showing that the principle minors all have a positive determinant.

6. [15] Consider the plane $\mathcal{W} \subset \mathbb{R}^4$ spanned by $\begin{pmatrix} 1 & 1 & -1 & -1 \end{pmatrix}^T$ and $\begin{pmatrix} 1 & -1 & 1 & -1 \end{pmatrix}^T$. Equip $\mathbb{R}^4$ with the Euclidean inner product.
   (a) Find the point in $\mathcal{W}$ that is closest to the point $\begin{pmatrix} 1 & 0 & 1 & 0 \end{pmatrix}^T$.
   (b) Give the shortest distance from $\mathcal{W}$ to the point $\begin{pmatrix} 1 & 0 & 1 & 0 \end{pmatrix}^T$.

Solution (a). Let

$$\mathbf{w}_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \qquad \mathbf{w}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \qquad \text{and} \qquad \mathbf{u} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

The point in $\mathcal{W}$ closest to $\mathbf{u}$ is given by the orthogonal projection of $\mathbf{u}$ onto $\mathcal{W}$. Because

$$\mathbf{w}_1^T\mathbf{w}_2 = 0\,, \qquad \mathbf{w}_1^T\mathbf{w}_1 = \mathbf{w}_2^T\mathbf{w}_2 = 4\,, \qquad \mathbf{w}_1^T\mathbf{u} = 0\,, \qquad \mathbf{w}_2^T\mathbf{u} = 2\,,$$

we see that $\{\mathbf{w}_1, \mathbf{w}_2\}$ is an orthogonal basis for $\mathcal{W}$ and that the orthogonal projection of $\mathbf{u}$ onto $\mathcal{W}$ is therefore given by

$$P\mathbf{u} = \frac{\mathbf{w}_1^T\mathbf{u}}{\mathbf{w}_1^T\mathbf{w}_1}\,\mathbf{w}_1 + \frac{\mathbf{w}_2^T\mathbf{u}}{\mathbf{w}_2^T\mathbf{w}_2}\,\mathbf{w}_2 = 0\,\mathbf{w}_1 + \tfrac{1}{2}\,\mathbf{w}_2 = \tfrac{1}{2}\,\mathbf{w}_2\,.$$

The point $\mathbf{w}$ in $\mathcal{W}$ that is closest to $\mathbf{u}$ is thereby

$$\mathbf{w} = P\mathbf{u} = \tfrac{1}{2}\,\mathbf{w}_2 = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}^T.$$

Alternative Solution (a). Let

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 1 \\ -1 & -1 \end{pmatrix}, \qquad \text{and} \qquad \mathbf{u} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

The point in $\mathcal{W}$ that is closest to $\mathbf{u}$ is given in terms of the least least squares solution with respect to the Euclidean inner product of the overdetermined system

$$A \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \mathbf{u}.$$

Specifically, the point $\mathbf{w}$ in $\mathcal{W}$ that is closest to $\mathbf{u}$ is given by

$$\mathbf{w} = A \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \qquad \text{where} \quad A^T A \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = A^T \mathbf{u}.$$

Because

$$A^T A = \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix},$$

$$A^T \mathbf{u} = \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$$

we see that

$$\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix},$$

whereby $c_1 = 0$ and $c_2 = \frac{1}{2}$. The point in $\mathcal{W}$ that is closest to $\mathbf{u}$ is thereby

$$\mathbf{w} = A \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}^T.$$

Solution (b). The shortest distance from $\mathcal{W}$ to $\mathbf{u}$ is $\|\mathbf{u} - \mathbf{w}\|$ where

$$\mathbf{u} - \mathbf{w} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}.$$

One thereby sees that this distance is

$$\|\mathbf{u} - \mathbf{w}\| = \sqrt{(\tfrac{1}{2})^2 + (\tfrac{1}{2})^2 + (\tfrac{1}{2})^2 + (\tfrac{1}{2})^2} = \sqrt{\tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{4}} = \sqrt{1} = 1.$$