**Second In-Class Exam Solutions**
**Math 220, Professor David Levermore**
**Friday, 29 October 2010**

(1) [24] Compute the first derivatives of the following functions.

(a) $f(x) = e^{\pi x^3} + \ln(1 + x^4)$

**Solution.** Using the exponential, logorithm, and power rules yields

$$f'(x) = e^{\pi x^3}\pi 3x^2 + \frac{1}{1+x^4}\,4x^3\,.$$

(b) $g(r) = \ln\big(e^{3r}(r+2)^5\big)$

**Solution.** First use $\ln(uv) = \ln(u) + \ln(v)$ to simplify $g(r)$ as

$$g(r) = \ln\big(e^{3r}\big) + \ln\big((r+2)^5\big) = 3r + 5\ln(r+2)\,.$$

Then differentiate using the power and logorithm rules to obtain

$$g'(r) = 3 + 5\,\frac{1}{r+2}\,.$$

(c) $h(s) = 5^{s^3}$

**Solution.** First express $h(s)$ in terms of the natural base as

$$h(s) = 5^{s^3} = \big(e^{\ln(5)}\big)^{s^3} = e^{\ln(5)s^3}\,.$$

Then differentiate using the exponential and power rules to obtain

$$h'(s) = e^{\ln(5)s^3}\,\ln(5)3s^2\,.$$

Had you simplifed, which was not required, this becomes

$$h'(s) = 5^{s^3}\,\ln(5)3s^2\,.$$

(d) $j(t) = \dfrac{\ln(1+t^2)}{t^2 - 4t + 5}$

**Solution.** By the quotient, logorithm, and power rules

$$j'(t) = \frac{\dfrac{1}{1+t^2}\,2t\,(t^2-4t+5) - \ln(1+t^2)\,(2t-4)}{(t^2-4t+5)^2}\,.$$

(2) [12] Solve the following equations for $x$.

(a) $\ln(x+1) - \ln(x-1) = 1$.

**Solution.** By the laws of logorithms this equation is equivalent to

$$\ln\left(\frac{x+1}{x-1}\right) = 1\,.$$

By exponentiating both sides of this equation you see that

$$\frac{x+1}{x-1} = e^1 = e\,.$$

This is solved for $x$ by first clearing the denominator to obtain the linear equation

$$x + 1 = e(x-1) = ex - e\,,$$

then grouping the $x$ terms on one side as

$$1 + e = ex - x = (e-1)x\,,$$

and finally solving for $x$ to find that

$$x = \frac{e+1}{e-1}\,.$$

(b) $\left(3^{x+1} \cdot 3^{-2}\right)^2 = 9$.

**Solution.** By taking the positive square root of both sides of the equation you obtain

$$3^{x+1} \cdot 3^{-2} = 3\,.$$

By the laws of exponents

$$3^{x+1} \cdot 3^{-2} = 3^{x+1-2} = 3^{x-1}\,,$$

whereby the equation reduces to $3^{x-1} = 3$. Therefore $x - 1 = 1$, which yields $x = 2$.

(3) [16] Consider the function $g(x) = x\ln(x) - x + 1$ over $x > 0$.

(a) Find the line tangent to the graph of $g$ at $x = e$.

**Solution.** Because $\ln(e) = 1$ you see that

$$g(e) = e\ln(e) - e + 1 = e \cdot 1 - e + 1 = e - e + 1 = 1\,.$$

The tangent line at $x = e$ therefore goes through the point $(e, 1)$.
By the product and logorithm rules one sees that

$$g'(x) = 1 \cdot \ln(x) + x \cdot \frac{1}{x} - 1 = \ln(x)\,.$$

Because $\ln(e) = 1$ you see that $g'(e) = \ln(e) = 1$. The tangent line at $x = e$ therefore has slope 1.
Because the tangent line at $x = e$ goes through the point $(e, 1)$ and has slope 1, it is given by the equation

$$y = 1 + 1 \cdot (x - e) = x - e + 1\,.$$

(b) Find the point $x$ where $g'(x) = 0$. Determine if it is a minimum or maximum.

**Solution.** Because $g'(x) = \ln(x)$, we see that $g'(x) = 0$ when $\ln(x) = 0$, which implies that

$$x = e^{\ln(x)} = e^0 = 1\,.$$

One way to show that $g(x)$ is minimum when $x = 1$ is to observe that $g'(x) = \ln(x) < 0$ when $0 < x < 1$, while $g'(x) = \ln(x) > 0$ when $1 < x < \infty$. By the First Derivative Test for Monotonicity $g(x)$ is decreasing when $0 < x < 1$ and increasing when $1 < x < \infty$. Therefore $g(x)$ is minimum at $x = 1$.
Another way to show that $g(x)$ is minimum when $x = 1$ is to observe that $g''(x) = 1/x > 0$ when $0 < x < \infty$. By the Second Derivative Test for Concavity $g(x)$ is concave up when $0 < x < \infty$. Therefore $g(x)$ is minimum at $x = 1$.

(4) [16] The price a company must charge in order to sell $q$ items is $p = e^{-\frac{1}{50}q}$.
(a) What is the company's revenue as a function of $q$?
(b) What value of $q$ will maximize its revenue?
(c) What is its marginal revenue as a function of $q$?
(d) What value of $q$ will minimize its marginal revenue?

**Solution (a).** Because revenue $R = qp$ where $p = e^{-\frac{1}{50}q}$, the revenue as a function of $q$ is given by

$$R(q) = qe^{-\frac{1}{50}q}\,.$$

**Solution (b).** To minimize $R(q)$ you first compute its derivative as

$$R'(q) = 1 \cdot e^{-\frac{1}{50}q} + qe^{-\frac{1}{50}q}\big(-\tfrac{1}{50}\big) = \left(1 - \frac{q}{50}\right) e^{-\frac{1}{50}q}\,.$$

Because $e^{-\frac{1}{50}q} > 0$ the only critical point of $R(q)$ is when $(1 - \frac{q}{50}) = 0$, which is when $q = 50$. Because $R'(q) > 0$ when $0 \leq q < 50$ while $R'(q) < 0$ when $50 < q$, by the First Derivative Monotonicity Test $R(q)$ is increasing when $0 \leq q < 50$ and is decreasing when $50 < q$. Therefore the revenue $R(q)$ is maximum when $q = 50$.

**Solution (c).** The marginal revenue as a function of $q$ is given by

$$R'(q) = 1 \cdot e^{-\frac{1}{50}q} + qe^{-\frac{1}{50}q}\big(-\tfrac{1}{50}\big) = \left(1 - \frac{q}{50}\right) e^{-\frac{1}{50}q}\,.$$

**Solution (d).** To minimize $R'(q)$ you first compute its derivative as

$$\begin{aligned} R''(q) &= -\frac{1}{50}\, e^{-\frac{1}{50}q} + \left(1 - \frac{q}{50}\right) e^{-\frac{1}{50}q}\big(-\tfrac{1}{50}\big) \\ &= -\tfrac{1}{50}\left(1 + 1 - \frac{q}{50}\right) e^{-\frac{1}{50}q} = -\tfrac{1}{50}\left(2 - \frac{q}{50}\right) e^{-\frac{1}{50}q}\,. \end{aligned}$$

Because $e^{-\frac{1}{50}q} > 0$ the only critical point of $R'(q)$ is when $(2 - \frac{q}{50}) = 0$, which is when $q = 100$. Because $R''(q) < 0$ when $0 \leq q < 100$ while $R''(q) > 0$ when $100 < q$, by the First Derivative Monotonicity Test $R'(q)$ is decreasing when $0 \leq q < 100$ and is increasing when $100 < q$. Therefore the marginal revenue $R'(q)$ is minimum when $q = 100$.

(5) [16] You wish to construct a box with a square base that has a volume of 10 cubic feet. If the cost of the material to make the bottom is 3 dollars per square foot while the cost of the material to make the sides and top is 2 dollars per square foot, find the dimensions of the box that minimize the total cost of the material to make it.

**Solution.** Let $x$ denote the length (in feet) of each edge along the base of a box. Let $h$ denote the height (in feet) of the box. The objective is to minimize the cost $C$ of the box. Because the top of the box has area $x^2$ while the bottom and four sides have total area $x^2 + 4xh$, the cost $C$ (in dollars) of the box is given by

$$C = 3x^2 + 2(x^2 + 4xh) = 3x^2 + 2x^2 + 8xh = 5x^2 + 8xh\,.$$

The volume of the box is constrained to be 10 cubic feet, which means

$$x^2 h = 10\,.$$

The simplest approach is to use the constraint to eliminate $h$ from the objective $C$. Because $h = 10/x^2$ you find that

$$C(x) = 5x^2 + 8x\,\frac{10}{x^2} = 5x^2 + \frac{80}{x}\,.$$

Because

$$C'(x) = 10x - \frac{80}{x^2} = \frac{10x^3 - 80}{x^2} = \frac{10(x^3 - 8)}{x^2}\,,$$

the critical points of $C(x)$ are where $x^3 = 8$, which is when the sides of the base have length $x = 2$. The associated height is then $h = 10/2^2 = 10/4 = 5/2$.

Because $C'(x) < 0$ when $0 < x < 2$ while $C'(x) > 0$ when $2 < x$, by the First Derivative Monotonicity Test we see that $C(x)$ is decreasing when $0 < x < 2$ and increasing when $2 < x$. Therefore the cost $C(x)$ is minimum when $x = 2$, whereby the dimensions of the box that minimize the cost are 2 feet $\times$ 2 feet $\times$ 2.5 feet.

(6) [16] A distributor of sporting goods expects to sell 500 boxes of baseball bats at a steady rate over the coming year. Yearly carrying costs are 10 dollars per box, while the cost of placing an order with the manufacturer is 25 dollars. How many orders should be placed to minimize the inventory cost?

**Solution.** Suppose that over the year we make $n$ orders of $b$ boxes each. The cost of making $n$ orders will be $25n$. Because there will be an average of $b/2$ boxes in the inventory over the year, the associated carrying cost will be $10b/2$. The inventory cost $C$ will therefore be

$$C = 25n + 10\frac{b}{2} = 25n + 5b\,.$$

The objective is to minimize $C$ subject to the constraint $nb = 500$.

Because you are being asked to find the $n$ that minimizes $C$, it is best to use the constraint to eliminate $b$ from $C$. Because $b = 500/n$, you find that

$$C(n) = 25n + 5\,\frac{500}{n} = 25n + \frac{2500}{n}\,.$$

Because

$$C'(n) = 25 - \frac{2500}{n^2} = \frac{25}{n^2}(n^2 - 100)\,,$$

the critical points of $C(n)$ are where $n^2 = 100$, which is when $n = 10$. (Then $b = 500/10 = 50$.)

Because $C'(n) < 0$ when $0 < n < 10$ while $C'(n) > 0$ when $10 < n$, by the First Derivative Monotonicity Test we see that $C(n)$ is decreasing when $0 < n < 10$ and is increasing when $10 < n$. Therefore the inventory cost $C(n)$ is minimum when 10 orders are placed.