Third In-Class Exam Solutions
Math 220, Professor David Levermore
Friday, 3 December 2010

(1) [18] Determine the indefinite integrals

(a) $\displaystyle\int \left(e^{3x} - 7x^3 + \frac{5}{x}\right) \mathrm{d}x$

Solution. Because

$$\int e^{3x}\,\mathrm{d}x = \frac{1}{3}e^{3x} + C\,, \qquad \int x^3\,\mathrm{d}x = \frac{1}{4}x^4 + C\,, \qquad \int \frac{1}{x}\,\mathrm{d}x = \ln(|x|) + C\,,$$

you see that

$$\int \left(e^{3x} - 7x^3 + \frac{5}{x}\right) \mathrm{d}x = \frac{1}{3}e^{3x} - \frac{7}{4}x^4 + 5\ln(|x|) + C\,.$$

(b) $\displaystyle\int \left(\sqrt[3]{x} + e^{3-x}\right) \mathrm{d}x$

Solution. Because

$$\int \sqrt[3]{x}\,\mathrm{d}x = \int x^{\frac{1}{3}}\,\mathrm{d}x = \frac{3}{4}x^{\frac{4}{3}} + C\,,$$

$$\int e^{3-x}\,\mathrm{d}x = e^3 \int e^{-x}\,\mathrm{d}x = e^3 \frac{1}{-1}e^{-x} + C = -e^{3-x} + C\,,$$

you see that

$$\int \left(\sqrt[3]{x} + e^{3-x}\right) \mathrm{d}x = \frac{3}{4}x^{\frac{4}{3}} - e^{3-x} + C\,.$$

(2) [15] Find the area of the region bounded by the curves $y = x^2$ and $y = 2x + 3$.

Solution. The curves intersect when $x^2 = 2x + 3$. By solving for x you find that

$$0 = x^2 - 2x - 3 = (x+1)(x-3)\,,$$

which means that $x = -1$ and $x = 3$. By checking $x = 0$ you see that the curve $y = 2x+3$ lies above the curve $y = x^2$. The area of the region bounded by the curves $y = x^2$ and $y = 2x + 3$ is therefore given by

$$\begin{aligned}
\int_{-1}^{3} 2x + 3 - x^2\,\mathrm{d}x &= \left(x^2 + 3x - \tfrac{1}{3}x^3\right)\Big|_{-1}^{3} \\
&= \left(3^2 + 3\cdot 3 - \frac{1}{3}3^3\right) - \left((-1)^2 + 3(-1) - \tfrac{1}{3}(-1)^3\right) \\
&= (9 + 9 - 9) - \left(1 - 3 + \tfrac{1}{3}\right) = \tfrac{32}{3}\,.
\end{aligned}$$

Full credit if your final answer is the second line above.

(3) [18] In this problem you do not have to evaluate any exponents or logarithms that occur in the answers.

(a) Find $\displaystyle\int_1^3 \left(t^3 + \frac{4}{t}\right) \mathrm{d}t$.

Solution. Because

$$\int t^3 \,\mathrm{d}t = \frac{1}{4}t^4 + C\,, \qquad \int \frac{1}{t} \,\mathrm{d}t = \ln(|t|) + C\,,$$

you see that

$$\begin{aligned}\int_1^3 \left(t^3 + \frac{4}{t}\right) \mathrm{d}t &= \left(\tfrac{1}{4}t^4 + 4\ln(|t|)\right)\Big|_1^3 \\ &= \left(\tfrac{1}{4}3^4 + 4\ln(3)\right) - \left(\tfrac{1}{4}1^4 + 4\ln(1)\right) \\ &= \left(\tfrac{81}{4} + 4\ln(3)\right) - \left(\tfrac{1}{4} + 4\cdot 0\right) = 20 + 4\ln(3)\,.\end{aligned}$$

Full credit if your final answer is the second line above.

(b) Find the average value of e^x between $x = 2$ and $x = 4$.

Solution. Because

$$\int e^x \,\mathrm{d}x = e^x + C\,,$$

you see that the average value of e^x between $x = 2$ and $x = 4$ is

$$\frac{1}{4-2}\int_2^4 e^x \,\mathrm{d}x = \frac{1}{2}e^x\Big|_2^4 = \frac{1}{2}e^4 - \frac{1}{2}e^2 = \frac{e^4 - e^2}{2}\,.$$

(4) [16] Money is deposited into a savings account that pays interest compounded continuously at a rate such that the balance of the account doubles every twenty years. (In this problem you do not have to evaluate any exponents or logarithms that occur in the answers.)

(a) What is the interest rate?
(b) Write a formula for $B(t)$, the balance after t years if the deposit is B_o.
(c) What is the differential equation satisfied by $B(t)$?
(d) If the deposit is 1000\$, what is $B(10)$?

Solution (a). The balance of the account is $B(t) = B_o e^{rt}$ where B_o is the initial deposit and r is the interest rate. You are told that $B(20) = B_o e^{r20} = 2B_o$, which means that r satisfies $e^{20r} = 2$. By solving this equation for r you find

$$r = \frac{\ln(2)}{20}\,.$$

If you chose to express this in percent then it is $r = 5\ln(2)\%$.

Solution (b). By the solution to (a) we see that

$$B(t) = B_o e^{\frac{\ln(2)}{20}t} = B_o 2^{\frac{t}{20}}\,.$$

Either form of the answer is fine.

Solution (c). In general, the balance $B(t)$ of an account that pays continuously compounded interest at rate r satisfies the differential equation $B'(t) = rB(t)$. By part (a) $r = \frac{\ln(2)}{20}$, so $B(t)$ satisfies the differential equation

$$B'(t) = \tfrac{\ln(2)}{20} B(t) .$$

Solution (d). By the solution to (b) with $B_o = 1000$ we see that

$$B(10) = 1000e^{\frac{\ln(2)}{20}10} = 1000e^{\frac{\ln(2)}{2}} = 1000 \cdot 2^{\frac{1}{2}} = 1000\sqrt{2} .$$

Any of the above forms of the answer is fine.

(5) [18] Consider the function $h(x, y) = x^2 - y^3 - 6x + 12y$ in the following.
- (a) Give the equation of the level curve of $h(x, y)$ that contains the point $(2, -1)$.
- (b) Find all points (x, y) where the function $h(x, y)$ has a possible relative maximum or relative minimum. (You do not have to determine if these points are relative maximums or relative minimums.)

Solution (a). The equation of the level curve of $h(x, y)$ that contains the point $(2, -1)$ is $h(x, y) = h(2, -1)$. Because $h(x, y) = x^2 - y^3 - 6x + 12y$, you find that $h(2, -1) = 2^2 - (-1)^3 - 6 \cdot 2 + 12 \cdot (-1) = 4 + 1 - 12 - 12 = -19$, and that the equation of the level curve is given by

$$x^2 - y^3 - 6x + 12y = -19 .$$

Solution (b). Because $h(x, y) = x^2 - y^3 - 6x + 12y$, you find that

$$\frac{\partial h}{\partial x}(x, y) = 2x - 6 , \qquad \frac{\partial h}{\partial y}(x, y) = -3y^2 + 12 .$$

The points where $h(x, y)$ has a possible relative maximum or relative minimum are found by setting these partial derivatives to zero, — i.e. by setting

$$0 = 2x - 6 = 2(x - 3) , \qquad 0 = -3y^2 + 12 = -3(y^2 - 4) = -3(y + 2)(y - 2) .$$

The solution of this system of equations is $x = 3$ and $y = \pm 2$. The points (x, y) where $h(x, y)$ has a possible relative maximum or relative minimum are therefore $(3, -2)$ and $(3, 2)$.

(6) [15] Let $g(x, y) = (5x + y^2)^3$. Find $\dfrac{\partial g}{\partial x}$, $\dfrac{\partial^2 g}{\partial x^2}$, and $\dfrac{\partial^2 g}{\partial x \partial y}$.

Solution. Because $g(x, y) = (5x + y^2)^3$, we find that

$$\begin{aligned}
\frac{\partial g}{\partial x}(x, y) &= 3(5x + y^2)^2 5 = 15(5x + y^2)^2 , \\
\frac{\partial^2 g}{\partial x^2}(x, y) &= \frac{\partial}{\partial x}\Big(15(5x + y^2)^2\Big) = 15 \cdot 2(5x + y^2)5 = 150(5x + y^2) , \\
\frac{\partial^2 g}{\partial x \partial y}(x, y) &= \frac{\partial^2 g}{\partial y \partial x}(x, y) \\
&= \frac{\partial}{\partial y}\Big(15(5x + y^2)^2\Big) = 15 \cdot 2(5x + y^2)2y = 60y(5x + y^2) .
\end{aligned}$$