1. [10] Let $X$ be a field. Use the field axioms to show that if $x, y \in X$ such that $xy = 1$ then $y = x^{-1}$.

**Remark.** The main point to keep in mind when doing problems like this is *to justify every step in your solution either by one or more of the axioms or by a previous step.*

**Solution.** First $x \neq 0$ because if $x = 0$ then $1 = xy = 0y = 0$, which is a contradiction. (The fact that $0y = 0$ for every $y \in X$ was shown in class.) Then

\[
y = y \cdot 1 \quad \text{(mult. ident. axiom)}
\]
\[
= 1 \cdot y \quad \text{(mult. comm. axiom)}
\]
\[
= (x x^{-1}) y \quad \text{(mult. inv. axiom and } x \neq 0)\]
\[
= (x^{-1} x) y \quad \text{(mult. comm. axiom)}
\]
\[
= x^{-1} (xy) \quad \text{(mult. assoc. axiom)}
\]
\[
= x^{-1} 1 \quad \text{(because } xy = 1)\]
\[
= x^{-1} \quad \text{(mult. ident. axiom)}
\]

**Remark.** For completeness, here is the proof that $0y = 0$. Let $y \in X$. The additive identity axiom implies $0 = 0 + 0$. The distributive axiom then gives the equality

\[
y0 = y(0 + 0) = y0 + y0.
\]

Then

\[
0 = y0 + (-(y0)) \quad \text{(add. inv. axiom)}
\]
\[
= (y0 + y0) + (-(y0)) \quad \text{(above equality)}
\]
\[
= y0 + (y0 + (-(y0))) \quad \text{(add. assoc. axiom)}
\]
\[
= y0 + 0 \quad \text{(add. inv. axiom)}
\]
\[
= y0 \quad \text{(add. indent. axiom)}
\]
\[
= 0y \quad \text{(add. comm. axiom)}.
\]
2. [15] Give a counterexample to each of the following false assertions.
   (a) If a sequence \( \{a_k\}_{k \in \mathbb{N}} \) in \( \mathbb{R} \) is divergent then the subsequence \( \{a_{2k}\}_{k \in \mathbb{N}} \) is divergent.
   (b) A countable intersection of nested nonempty open intervals is also nonempty.
   (c) If \( \lim_{k \to \infty} a_k = 0 \) then \( \sum_{k=0}^{\infty} a_k \) converges.

Solution (a). A simple counterexample is obtained by setting \( a_k = (-1)^k \) because \( \{(-1)^k\} \) diverges, while \( \lim_{k \to \infty} (-1)^{2k} = 1 \).

Solution (b). A countable intersection of nested nonempty open intervals must have the form
\[
\bigcap_{k=0}^{\infty} (a_k, b_k)
\]
where \( a_k < b_k \) and \( (a_{k+1}, b_{k+1}) \subset (a_k, b_k) \) for every \( k \in \mathbb{N} \). Such an intersection that is empty is obtained by setting \( a_k = 0 \) and \( b_k = 2^{-k} \) for every \( k \in \mathbb{N} \).

Solution (c). A simple counterexample is obtained by setting \( a_k = 1/(k + 1) \) because
\[
\lim_{k \to \infty} \frac{1}{k + 1} = 0, \quad \text{while the harmonic series} \ \sum_{k=0}^{\infty} \frac{1}{k + 1} \ \text{diverges}.
\]

3. [15] Consider the real sequence \( \{c_k\}_{k \in \mathbb{N}} \) given by
\[
c_k = (-1)^k \frac{k + 3}{2k + 2} \quad \text{for every} \ k \in \mathbb{N},
\]
with the convention that \( \mathbb{N} = \{0, 1, 2, \ldots\} \).

(a) Write down the first three terms of the subsequence \( \{c_{2k+1}\}_{k \in \mathbb{N}} \).
(b) Write down \( \liminf_{k \to \infty} c_k \) and \( \limsup_{k \to \infty} c_k \). (No proof is needed here.)

Solution (a). You are given that \( \mathbb{N} = \{0, 1, 2, \ldots\} \), as was the convention in class and in the notes (but not in the book). The first three terms of the subsequence \( \{c_{2k+1}\}_{k \in \mathbb{N}} \) are therefore
\[
c_1 = -\frac{4}{4} = -1, \quad c_3 = -\frac{6}{8} = -\frac{3}{4}, \quad c_5 = -\frac{8}{12} = -\frac{2}{3}.
\]

Solution (b). Because \( c_{2k} > 0 \) while \( c_{2k+1} < 0 \), and because
\[
\lim_{k \to \infty} c_{2k} = \lim_{k \to \infty} \frac{2k + 3}{4k + 2} = \frac{1}{2},
\]
while
\[
\lim_{k \to \infty} c_{2k+1} = -\lim_{k \to \infty} \frac{2k + 4}{4k + 4} = -\frac{1}{2},
\]
one has that
\[
\limsup_{k \to \infty} c_k = \frac{1}{2}, \quad \liminf_{k \to \infty} c_k = -\frac{1}{2}.
\]
4. Let \( \{a_k\}_{k \in \mathbb{N}} \) and \( \{b_k\}_{k \in \mathbb{N}} \) be bounded, positive sequences in \( \mathbb{R} \).

(a) Prove that
\[
\limsup_{k \to \infty} (a_k b_k) \leq \left( \limsup_{k \to \infty} a_k \right) \left( \limsup_{k \to \infty} b_k \right).
\]

(b) Write down an example for which equality does not hold above.

Solution (a). Let \( c_k = a_k b_k \) for every \( k \in \mathbb{N} \). For every \( k \in \mathbb{N} \) define
\[
\overline{a}_k = \sup \{a_l : l \geq k\}, \quad \overline{b}_k = \sup \{b_l : l \geq k\}, \quad \overline{c}_k = \sup \{c_l : l \geq k\}.
\]
Because the sequences \( \{a_k\} \), \( \{b_k\} \), and \( \{c_k\} \) are positive and bounded above, the sequences \( \overline{a}_k \), \( \overline{b}_k \), and \( \overline{c}_k \), are positive and nonincreasing. The Monotonic Sequence Theorem thereby implies that the sequences \( \{a_k\} \), \( \{b_k\} \), and \( \{c_k\} \) are convergent. By the definition of \( \limsup \) we then have
\[
\limsup_{k \to \infty} a_k = \lim_{k \to \infty} a_k, \quad \limsup_{k \to \infty} b_k = \lim_{k \to \infty} b_k, \quad \limsup_{k \to \infty} c_k = \lim_{k \to \infty} c_k.
\]
The key observation is that for every \( k \in \mathbb{N} \) we have
\[
c_l = a_l b_l \leq \overline{a}_k \overline{b}_k \quad \text{for every } l \geq k,
\]
which yields the inequality
\[
\overline{c}_k = \sup \{c_l : l \geq k\} \leq \overline{a}_k \overline{b}_k.
\]
This inequality and the properties of limits then implies
\[
\limsup_{k \to \infty} (a_k b_k) = \limsup_{k \to \infty} c_k
\]
\[
= \lim_{k \to \infty} c_k
\]
\[
\leq \lim_{k \to \infty} (\overline{a}_k \overline{b}_k)
\]
\[
= \left( \lim_{k \to \infty} \overline{a}_k \right) \left( \lim_{k \to \infty} \overline{b}_k \right)
\]
\[
= \left( \limsup_{k \to \infty} a_k \right) \left( \limsup_{k \to \infty} b_k \right).
\]

Solution (b). Let \( a_k = 2^{(-1)^k} \) and \( b_k = 2^{(-1)^{k+1}} \) for every \( k \in \mathbb{N} \). Clearly
\[
\limsup_{k \to \infty} a_k = \lim_{k \to \infty} a_{2k} = 2, \quad \limsup_{k \to \infty} b_k = \lim_{k \to \infty} b_{2k+1} = 2,
\]
while (because \( a_k b_k = 1 \) for every \( k \in \mathbb{N} \))
\[
\limsup_{k \to \infty} (a_k b_k) = \lim_{k \to \infty} (a_k b_k) = 1.
\]
Therefore
\[
\limsup_{k \to \infty} (a_k b_k) = 1 < 4 = 2 \cdot 2 = \left( \limsup_{k \to \infty} a_k \right) \left( \limsup_{k \to \infty} b_k \right).
\]
5. [10] Let $X^c$ denote the closure of any subset $X$ of $\mathbb{R}$. Let $A$ and $B$ be subsets of $\mathbb{R}$. Prove that $A^c \cup B^c \subset (A \cup B)^c$.

Remark. You must show that every element of $A^c \cup B^c$ is also an element of $(A \cup B)^c$. If your proof directly uses the definition of closure then its first step should be clear.

Solution. Let $x \in A^c \cup B^c$ be arbitrary. Then either $x \in A^c$ or $x \in B^c$. (Both can be true.) Without loss of generality we can assume that $x \in A^c$. By the definition of closure, there exists a sequence $\{x_k\}_{k \in \mathbb{N}}$ contained in $A$ such that $x_k \to x$ as $k \to \infty$. But the sequence $\{x_k\}_{k \in \mathbb{N}}$ is therefore contained in both $A \cup B$ while $x_k \to x$ as $k \to \infty$. By the definition of closure, it follows that $x \in (A \cup B)^c$. But because $x \in A^c \cup B^c$ was arbitrary, we conclude that $A^c \cup B^c \subset (A \cup B)^c$.

Remark. You could also have built a proof around the fact that if $C \subset D$ then their closures satisfy $C^c \subset D^c$. (This is a fact you should be able to prove directly from the definition of closure.)

Alternative Solution. Because $A \subset (A \cup B)$ and $B \subset (A \cup B)$, we know $A^c \subset (A \cup B)^c$ and $B^c \subset (A \cup B)^c$. We conclude that $A^c \cup B^c \subset (A \cup B)^c$.

6. [15] Determine all $a \in \mathbb{R}$ for which
$$\sum_{k=0}^{\infty} \left( \frac{k^2 + 1}{k^4 + 1} \right)^a$$
converges.

Give your reasoning.

Solution. The series converges for $a \in \left( \frac{1}{2}, \infty \right)$ and diverges otherwise. Because
$$\frac{k^2 + 1}{k^4 + 1} \sim \frac{1}{k^2} \quad \text{as} \quad k \to \infty,$$
one sees that the original series should be compared with the $p$-series
$$\sum_{k=1}^{\infty} \frac{1}{k^{2a}}.$$

This is best handled by Limit Comparison Test. Indeed, because for every $a \in \mathbb{R}$ one has
$$\lim_{k \to \infty} \frac{\left( \frac{k^2 + 1}{k^4 + 1} \right)^a}{\frac{1}{k^{2a}}} = \lim_{k \to \infty} \left( \frac{k^4 + k^2}{k^4 + 1} \right)^a = 1,$$
the Limit Comparison Test then implies that
$$\sum_{k=0}^{\infty} \left( \frac{k^2 + 1}{k^4 + 1} \right)^a \text{ converges } \iff \sum_{k=1}^{\infty} \frac{1}{k^{2a}} \text{ converges}.$$

Because the $p = 2a$ for the $p$-series, it converges for $a \in \left( \frac{1}{2}, \infty \right)$ and diverges otherwise. The same is therefore true for the original series.
7. [10] Let \( \{a_k\}_{k \in \mathbb{N}} \) be a real sequence and \( \{a_{n_k}\} \) be any subsequence of it. Show that \( \sum_{k=0}^{\infty} a_k \) converges absolutely \( \implies \sum_{k=0}^{\infty} a_{n_k} \) converges absolutely.

**Solution.** By the definition of absolute convergence
\[
\sum_{k=0}^{\infty} a_k \text{ converges absolutely} \iff \sum_{k=0}^{\infty} |a_k| \text{ converges},
\]
\[
\sum_{k=0}^{\infty} a_{n_k} \text{ converges absolutely} \iff \sum_{k=0}^{\infty} |a_{n_k}| \text{ converges}.
\]
For every \( m, n \in \mathbb{N} \) define the sequences \( \{p_m\} \) and \( \{q_n\} \) of partial sums
\[
p_m = \sum_{k=0}^{m} |a_{n_k}|, \quad q_n = \sum_{k=0}^{n} |a_k|.
\]
It is clear that these sequences are nondecreasing and that the Monotonic Sequence Theorem implies
\[
\sum_{k=0}^{\infty} |a_k| \text{ converges} \iff \{q_n\} \text{ is bounded above},
\]
\[
\sum_{k=0}^{\infty} |a_{n_k}| \text{ converges} \iff \{p_m\} \text{ is bounded above}.
\]
Moreover \( p_m \) and \( q_n \) satisfy the inequality
\[
p_m = \sum_{k=0}^{m} |a_{n_k}| \leq \sum_{k=0}^{n_m} |a_k| = q_{n_m} \quad \text{for every } m \in \mathbb{N}.
\]
This inequality shows that if \( \{q_n\} \) is bounded above then \( \{p_m\} \) is bounded above. Hence,
\[
\sum_{k=0}^{\infty} a_k \text{ converges absolutely} \iff \{q_n\} \text{ is bounded above} \implies \{p_m\} \text{ is bounded above} \implies \sum_{k=0}^{\infty} a_{n_k} \text{ converges absolutely}.
\]

8. [10] Let \( \{b_k\}_{k \in \mathbb{N}} \) be a sequence in \( \mathbb{R} \) and let \( A \) be a subset of \( \mathbb{R} \). Write the negations of the following assertions.
(a) “There exists \( m \in \mathbb{R} \) such that \( b_j > m \) eventually as \( j \to \infty \).”
(b) “Every sequence in \( A \) has a subsequence that converges to a limit in \( A \).”

**Solution (a).** “For every \( m \in \mathbb{R} \) one has \( b_j \leq m \) frequently as \( j \to \infty \).”

**Solution (b).** “There is a sequence in \( A \) such that every subsequence of it either diverges or converges to a limit outside \( A \).”

**Remark.** The answer “There is a sequence in \( A \) such that no subsequence of it converges to a limit in \( A \)” does not fully carry the negation through.