

Sample Problems for First In-Class Exam
Math 246, Spring 2011, Professor David Levermore

- (1) (a) Give the integral being evaluated by the MATLAB command

$$\text{int('x/(1+x^4)', 'x', 0, \text{inf})}.$$

- (b) Sketch the graph that you expect would be produced by the following MATLAB commands.

```
[x, y] = meshgrid(-5:0.5:5, -5:0.5:5)
contour(x, y, x.^2 + y.^2, [1, 9, 25])
axis square
```

- (2) Find the explicit solution for each of the following initial-value problems and identify its interval of definition (existence).

(a) $\frac{dz}{dt} = \frac{\cos(t) - z}{1 + t}, \quad z(0) = 2.$

(b) $\frac{du}{dz} = e^u + 1, \quad u(0) = 0.$

- (3) Consider the differential equation

$$\frac{dy}{dt} = 4y^2 - y^4.$$

- (a) Find all of its stationary (equilibrium) solutions and classify each as being either stable, unstable, or semistable.
 (b) If $y(0) = 1$, how does the solution $y(t)$ behave as $t \rightarrow \infty$?
 (c) If $y(0) = -1$, how does the solution $y(t)$ behave as $t \rightarrow \infty$?
 (d) Sketch a graph of y versus t showing the direction field and several solution curves. The graph should show all the stationary solutions as well as solution curves above and below each of them. Every value of y should lie on at least one sketched solution curve.

- (4) A tank initially contains 100 liters of pure water. Beginning at time $t = 0$ brine (salt water) with a salt concentration of 2 grams per liter (g/l) flows into the tank at a constant rate of 3 liters per minute (l/min) and the well-stirred mixture flows out of the tank at the same rate. Let $S(t)$ denote the mass (g) of salt in the tank at time $t \geq 0$.

- (a) Write down an initial-value problem that governs $S(t)$.
 (b) Is $S(t)$ an increasing or decreasing function of t ? (Give your reasoning.)
 (c) What is the behavior of $S(t)$ as $t \rightarrow \infty$? (Give your reasoning.)
 (d) Derive an explicit formula for $S(t)$.

- (5) Suppose you are using the Heun-midpoint method to numerically approximate the solution of an initial-value problem over the time interval $[0, 5]$. By what factor would you expect the error to decrease when you increase the number of time steps taken from 500 to 2000?
- (6) Give an implicit general solution to each of the following differential equations.
- $\left(\frac{y}{x} + 3x\right) dx + (\log(x) - y) dy = 0.$
 - $(x^2 + y^3 + 2x) dx + 3y^2 dy = 0.$
- (7) A 2 kilogram (kg) mass initially at rest is dropped in a medium that offers a resistance of $v^2/40$ newtons ($= \text{kg m/sec}^2$) where v is the downward velocity (m/sec) of the mass. The gravitational acceleration is 9.8 m/sec^2 .
- What is the terminal velocity of the mass?
 - Write down an initial-value problem that governs v as a function of time. (You do not have to solve it!)
- (8) Consider the following MATLAB function M-file.

```
function [t,y] = solveit(ti, yi, tf, n)

h = (tf - ti)/n;
t = zeros(n + 1, 1);
y = zeros(n + 1, 1);
t(1) = ti;
y(1) = yi;
for i = 1:n
z = t(i)^4 + y(i)^2;
t(i + 1) = t(i) + h;
y(i + 1) = y(i) + (h/2)*(z + t(i + 1)^4 + (y(i) + h*z)^2);
end
```

- What is the initial-value problem being approximated numerically?
- What is the numerical method being used?
- What are the output values of $t(2)$ and $y(2)$ that you would expect for input values of $t_i = 1$, $y_i = 1$, $t_f = 5$, $n = 20$?