Quiz 5 Solutions, Math 246, Professor David Levermore
Tuesday, 8 March 2011

(1) [3] Show that e^{-t} and e^{2t} are linearly independent.

Solution. The Wronskian of e^{-t} and e^{2t} is

$$W[e^{-t}, e^{2t}](t) = \det\begin{pmatrix} e^{-t} & e^{2t} \\ -e^{-t} & 2e^{2t} \end{pmatrix}$$
$$= e^{-t}2e^{2t} - (-e^{-t})e^{2t} = 2e^{t} + e^{t} = 3e^{t}\,.$$

Because $W[e^{-t}, e^{2t}](t) = 3e^{t} \neq 0$, the functions e^{-t} and e^{2t} are linearly independent.

Alternative Solution. Suppose that

$$c_1 e^{-t} + c_2 e^{2t} = 0\,.$$

By setting $t = 0$ and $t = 1$ in this equation you obtain

$$c_1 + c_2 = 0\,, \qquad c_1 e^{-1} + c_2 e^{2} = 0\,.$$

This linear algebraic system can be solved by elimination to show that $c_1 = c_2 = 0$. Therefore the functions e^{-t} and e^{2t} are linearly independent.

(2) [7] Give a general solution of the equation

$$(\mathrm{D}+3)^2(\mathrm{D}-2)(\mathrm{D}^2+2\mathrm{D}+10)^2 y = 0\,, \qquad \text{where} \quad \mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}t}\,.$$

Solution. This is a seventh-order, homogeneous, linear differential equation with constant coefficients. Its characteristic polynomial is

$$p(z) = (z+3)^2(z-2)(z^2+2z+10)^2$$
$$= (z+3)^2(z-2)\big((z+1)^2+3^2\big)^2\,,$$

which has roots -3, -3, 2, $-1+i3$, $-1+i3$, $-1-i3$, and $-1-i3$. A general solution of the differential equation is

$$y(t) = c_1 e^{-3t} + c_2 t\,e^{-3t} + c_3 e^{2t} + c_4 e^{-t}\cos(3t) + c_5 e^{-t}\sin(3t)$$
$$+ c_6 t\,e^{-t}\cos(3t) + c_7 t\,e^{-t}\sin(3t)\,.$$

The reasoning is as follows:

- the double real root -3 yields the solutions e^{-3t} and $t\,e^{-3t}$;
- the simple real root 2 yields the solution e^{2t};
- the double conjugate pair $-1 \pm i3$ yields the solutions

$$e^{-t}\cos(3t)\,, \quad e^{-t}\sin(3t)\,, \quad t\,e^{-t}\cos(3t)\,, \quad \text{and} \quad t\,e^{-t}\sin(3t)\,.$$