Bose-Einstein Condensation and the Kompaneets Equation

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1. Photons within a Homogeneous Plasma

Photons can play a mojor role in the transport of energy within a fully ionized plasma through the processes of emission, absorpsion, and scattering. At high temperatures the domiant process can be Compton scattering off free electrons. We will consider a model of only this phenomena for a spatially homogeneous plasma at a fixed uniform temperature T.

Photons are generally described by a kinetic density $f(k, q, t)$ where k is the wave vector, q is the position, and t is time. (We will neglect polarization.) The phase space volume measure is $\mathrm{d}k\mathrm{d}q$. The momentum and energy of a photon with wave vector k are $\hbar k$ and $\hbar|k|$ respectively. We will assume the photon kinetic density is spatially uniform and isotropic in k. It thereby simplifies to the form $f(|k|, t)$.

We introduce the nondimensional radial variable $x = \hbar|k|/\mathsf{k}_{\mathsf{B}}T$, so that $f(x,t)$. The number and energy of the photon field is then

$$N[f] = \int_0^\infty f\, x^2 \,\mathrm{d}x\,, \qquad E[f] = \int_0^\infty f\, x^3 \,\mathrm{d}x\,.$$

In this setting the evolution of f due to Compton scattering is governed by a quantum kinetic equation of the form

$$\partial_t f = \int_0^\infty \sigma(x,x')\Big(e^{x'} f'(1+f) - e^{x} f(1+f')\Big)\, x'^2 \,\mathrm{d}x'\,,$$

where f' denotes $f(x',t)$. Here σ satifies the detailed balance relation $\sigma(x,x') = \sigma(x',x)$ over $\mathbb{R}_+ \times \mathbb{R}_+$. The first term represents photons whose energy is changed from x' to x by collision, while the second represents photons whose energy is changed from x' to x. The factors $(1+f)$ and $(1+f')$ arise because photons are Bosons, which means they prefer states already occupied by other photons.

2. Kompaneets Equation

Because the energy exchange of each collision is small, the Fokker-Planck approximation can be applied to the quantum Boltzmann equation. In non-relativistic regimes this yields the so-called *Kompaneets equation*,

$$\partial_t f = \frac{1}{x^2}\partial_x\Big[x^4(\partial_x f + f + f^2)\Big]\,, \qquad f\Big|_{t=0} = f^{\text{in}}\,. \tag{1}$$

Because x is a radial variable, here the divergence has the form $x^{-2}\partial_x x^2$, while the diffusion coefficient is x^2. Notice that the diffusion coefficent becomes degenerate (vanishes) at the origin!

The question arises as to whether or not a boundary condition needs to be imposed either at $x = 0$ and as $x \to \infty$. The answer to this question will depend upon the space in which one wants to establish well-posedness! For now we will proceed formally.

When we consider the evolution of $N[f]$ we find

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}N[f] &= \int_0^\infty \partial_t f \, x^2 \, \mathrm{d}x \\
&= \int_0^\infty \partial_x \Big[x^4(\partial_x f + f + f^2)\Big] \, \mathrm{d}x \\
&= \Big[x^4(\partial_x f + f + f^2)\Big]\Big|_0^\infty .
\end{aligned}$$

Because the Kompaneets equation (1) describes the evolution of f due to only a scattering process, the number of photons must be conserved. We therefore might expect that f should satisfy

$$\Big[x^4(\partial_x f + f + f^2)\Big]\Big|_{x=0} = \Big[x^4(\partial_x f + f + f^2)\Big]\Big|_{x\to\infty} = 0 .$$

Similarly, when we consider the evolution of $E[f]$ we find

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E[f] &= \int_0^\infty \partial_t f\, x^3 \,\mathrm{d}x \\ &= \int_0^\infty x\partial_x\Big[x^4(\partial_x f + f + f^2)\Big]\,\mathrm{d}x \\ &= \Big[x^5(\partial_x f + f + f^2)\Big]\Big|_0^\infty - \int_0^\infty x^4(\partial_x f + f + f^2)\,\mathrm{d}x\,.\end{aligned}$$

If we expect that energy does not “escape to infinity” then f should also satisfy

$$\Big[x^5(\partial_x f + f + f^2)\Big]\Big|_{x\to\infty} = 0\,.$$

3. Bose-Einstein Equilibria

Stationary solutions of the Kompaneets equation (1) satisfy

$$\partial_x \Big[x^4 (\partial_x f + f + f^2) \Big] = 0 \,.$$

The boundary behavior we have already assumed implies that

$$\partial_x f + f + f^2 = 0 \,.$$

Stationary solutions of the Kompaneets equation therefore have the form

$$f = f_\mu(x) = \frac{1}{e^{x+\mu} - 1} \quad \text{for some } \mu \geq 0 \,.$$

These comprise the family of Bose-Einstein equilibria. The Planckian is recovered when $\mu = 0$.

It is natural to ask whether every solution of the Kompaneets equation approaches one of these equilibria as $t \to \infty$. If so, because $N[f]$ is conserved, it would be natural to think that the solution associated with initial data f^{in} would approach f_μ determined by

$$N[f^{\mathrm{in}}] = N[f_\mu] = \int_0^\infty \frac{1}{e^{x+\mu} - 1} x^2 \mathrm{d}x \,.$$

But $N[f_\mu]$ is a strictly decreasing function of μ bounded above by $N[f_0]$! This picture is therefore clearly **WRONG** for any f^{in} with $N[f^{\mathrm{in}}] > N[f_0]$, and might even be wrong for some f^{in} with $N[f^{\mathrm{in}}] \leq N[f_0]$.

What went wrong? Do some f not approach an equilibrium? If so, do they have dynamic long-time behavior or do they become singular? Are our assumptions about the boundary behavior wrong? If so, is $N[f]$ conserved?

4. Quantum Entropy Structure

Another physical quantity is the quantum entropy (Helmholtz free energy)

$$H[f] = \int_0^\infty h(f,x)\, x^2 \, \mathrm{d}x \,,$$

where the entropy density is given by

$$h(f,x) = f \log(f) - (1+f)\log(1+f) + xf \,.$$

Because the partial derivatives of $h(f,x)$ with respect to f are

$$h_f(f,x) = \log(f) - \log(1+f) + x = \log\left(\frac{e^x f}{1+f}\right),$$

$$h_{ff}(f,x) = \frac{1}{f} - \frac{1}{1+f} = \frac{1}{f(1+f)}\,,$$

we see that $H[f]$ is a strictly convex function of f.

Because $\dfrac{\delta H[f]}{\delta f} = h_f(f,x)$ and

$$\partial_x h_f = h_{ff}\partial_x f + 1 = \frac{1}{f(1+f)}(\partial_x f + f + f^2)\,,$$

we see that the Kompaneets equation (1) can be expressed as

$$\partial_t f = \frac{1}{x^2}\partial_x\left[x^4 f(1+f)\partial_x \frac{\delta H[f]}{\delta f}\right].$$

Except for the factor $(1+f)$, this looks like the usual optimal transportion equation associated with the gradient-flow of $H[f]$! The factor $(1+f)$ as well as the form of $h(f,x)$ arises from the underlying Bose-Einstein statistics. In other words, they arise from quantum mechanics.

Remark. In the Fermi-Dirac analog the factor $(1+f)$ becomes $(1-f)$ while the entropy density becomes

$$h(f,x) = f\log(f) + (1-f)\log(1-f) + xf\,.$$

When we consider the evolution of $H[f]$ we find that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}H[f] &= \int_0^\infty h_f\,\partial_t f\,x^2\,\mathrm{d}x \\
&= \int_0^\infty h_f\,\partial_x\Big[x^4 f(1+f)\partial_x h_f\Big]\,\mathrm{d}x \\
&= \Big(h_f\Big[x^4 f(1+f)\partial_x h_f\Big]\Big)\Big|_0^\infty - \int_0^\infty x^4 f(1+f)(\partial_x h_f)^2\,\mathrm{d}x\,.
\end{aligned}$$

We might expect that

$$h_f\Big[x^4 f(1+f)\partial_x h_f\Big]\Big|_{x=0} = h_f\Big[x^4 f(1+f)\partial_x h_f\Big]\Big|_{x\to\infty} = 0\,.$$

When that is the case we obtain the entropy dissipation relation

$$\frac{\mathrm{d}}{\mathrm{d}t}H[f] = -\int_0^\infty x^4 f(1+f)(\partial_x h_f)^2\,\mathrm{d}x\,.$$

This dissipation relation would rule out long-time behavior that is either periodic, quasi-periodic, or almost periodic.

Moreover, for $f > 0$ one sees that the dissipation vanishes if and only if

$$\partial_x h_f = \partial_x \log\left(\frac{e^x f}{1+f}\right) = 0\,,$$

which is if and only if

$$f = f_\mu(x) = \frac{1}{e^{x+\mu} - 1} \quad \text{for some } \mu \geq 0\,.$$

This is an analog of the Boltzmann H-Theorem.

However, these assertions can be called into question if the above formal calculation is not justified and $H[f]$ does not dissipate.

5. Classical Analog

The analog of the Kompaneets equation for classical statistics is

$$\partial_t f = \frac{1}{x^2}\partial_x\Big[x^4(\partial_x f + f)\Big]\,, \qquad f\Big|_{t=0} = f^{\text{in}}\,.$$

Because this equation is linear, it can be understood much better than the Kompaneets equation. Its associated entropy density is

$$H[f] = \int_0^1 h(f,x)\,x^2 \mathrm{d}x\,, \qquad \text{where} \quad h(f,x) = f\log(f) - f + xf\,.$$

It has the optimal transportation form

$$\partial_t f = \frac{1}{x^2}\partial_x\left[x^4 f \partial_x \frac{\delta H[f]}{\delta f}\right],$$

where

$$\frac{\delta H[f]}{\delta f} = h_f(f,x) = \log(f) + x = \log(e^x f)\,.$$

Its family of equilibria is

$$f_\mu(x) = e^{-x-\mu} \qquad \text{for } \mu \in \mathbb{R}\,.$$

One can show that this linear initial-value problem is well-posed in the cone of nonnegative densities f such that

$$\int_0^\infty (e^x f)^p e^{-x} x^2 \mathrm{d}x < \infty \qquad \text{for some } p \in (1,\infty)\,,$$

or such that $H[f] < \infty$. These solutions

- are smooth over $\mathbb{R}_+ \times \mathbb{R}_+$,
- are positive over $\mathbb{R}_+ \times \mathbb{R}_+$ provided $f^{\text{in}} \neq 0$,
- satisfy all the expected boundary conditions ,
- conserve $N[f]$ and dissipate $H[f]$ as expected ,
- approach f_μ as $t \to \infty$ where $N[f^{\text{in}}] = N[f_\mu]$.

In particular, no boundary condition needs to be imposed!

6. What is Happening: Bose-Einstein Condensation

The problem clearly arises from the quadratic quantum term. To see what might be happening, let's keep only that term. The Kompaneets equation then becomes

$$\partial_t f = \frac{1}{x^2}\partial_x\left[x^4 f^2\right], \qquad f\Big|_{t=0} = f^{\mathrm{in}}.$$

Introducing $u = x^2 f$, this becomes the inviscid Burgers equation

$$\partial_t u = \partial_x\left[u^2\right] = 2u\partial_x u, \qquad u\Big|_{t=0} = u^{\mathrm{in}},$$

where $u^{\mathrm{in}} = x^2 f^{\mathrm{in}}$. Its characteristic equations are

$$\dot{x} = -2u, \qquad \dot{u} = 0.$$

Because $u \geq 0$, no boundary condition is needed at the origin. Clearly any nonzero viscosity solution will develop a nonzero flux of photons into the origin in finite time. These photons cannot disappear! Rather, they produce a "delta function" at the origin — a *Bose-Einstein condensate*.

If we only drop the $\partial_x f$ term then the Kompaneets equation becomes

$$\partial_t f = \frac{1}{x^2}\partial_x\Big[x^4(f+f^2)\Big]\,, \qquad f\Big|_{t=0} = f^{\mathrm{in}}\,.$$

Again letting $u = x^2 f$ and $u^{\mathrm{in}} = x^2 f^{\mathrm{in}}$, this becomes

$$\partial_t u = \partial_x\Big[x^2 u + u^2\Big]\,, \qquad u\Big|_{t=0} = u^{\mathrm{in}}\,.$$

This has the characteristic equations

$$\dot{x} = -x^2 - 2u\,, \qquad \dot{u} = 2xu\,.$$

The addtional terms hasten the formation of the condensate.

The formation of a Bose-Einstein condensate means that the “expected” boundary behavior of the solution at the origin breaks down.

Previous analytic studies of the Kompaneets equation studied its classical solutions before the onset of the condensate:

- R. Caflisch and D.L. (1986)
- D.L. and O. Kavian (1990, unpublished)
- M. Escobedo, M.A. Herrero, and J.J.L. Velazquez (1998)

Approaches that treat the nonlinear term as a perturbation of the linear ones can show that solutions can “blows up” out of spaces associated with the linear problem, but are doomed to capture the condensate because functions in those spaces have the “expected” boundary behavior at the origin — which breaks down.

7. Results for a Model Problem

In order to better understand the phenomenon, H. Liu, R. Pego, and I have been studying the Kompaneets equation with the f term dropped, namely

$$\partial_t f = \frac{1}{x^2}\partial_x\Big[x^4(\partial_x f + f^2)\Big]\,, \qquad f\Big|_{t=0} = f^{\mathrm{in}}\,.$$

We do this over $x \in [0,1]$ and impose the boundary condition

$$(\partial_x f + f^2)\Big|_{x=1} = 0\,.$$

The equilibria for this equation are

$$f_\mu(x) = \frac{1}{x+\mu} \qquad \text{for some } \mu \geq 0\,.$$

We posed the problem on a bounded interval to capture these. The hope is that it will also capture the formation and evolution of condensates.

Again letting $u = x^2 f$ and $u^{\mathrm{in}} = x^2 f^{\mathrm{in}}$, our model becomes

$$\partial_t u = \partial_x \Big[x^2 \partial_x u - 2xu + u^2 \Big] , \qquad u\Big|_{t=0} = u^{\mathrm{in}} ,$$

considered over $[0,1]$ with the boundary condition

$$\Big[\partial_x u - 2u + u^2 \Big]\Big|_{x=1} = 0 .$$

The equilibria then have the form

$$u_\mu(x) = \frac{x^2}{x+\mu} \quad \text{for some } \mu \geq 0 .$$

The entropy for this equation is

$$H[u] = \int_0^1 \Big(xu - x^2 \log(u) \Big) \, \mathrm{d}x .$$

We consider initial data $u^{\mathrm{in}} \in L^1([0,1], \mathrm{d}x)$ such that $u^{\mathrm{in}}(x) > 0$ a.e. and $H[u^{\mathrm{in}}] < \infty$.

We show there exists a unique solution that satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{0^+}^{1} u \,\mathrm{d}x = -u^2\Big|_{x=0^+} .$$

The number of photons in the condensate at time t is therefore

$$\int_0^t u(0^+, s)^2 \mathrm{d}s .$$

Once a photon enters the condensate, it stays there.

We show that

$$\frac{\mathrm{d}}{\mathrm{d}t} H[u] \leq -\int_{0^+}^{1} u^2 \left(1 - \partial_x \left(\frac{x^2}{u}\right)\right)^2 \mathrm{d}x .$$

The entropy $H[u]$ does not see the Bose-Einstein condensate.

Our approach is to pass to the $\epsilon \to 0$ limit in the weak formulation of the regularized problem

$$\partial_t u_\epsilon = \partial_x \left[x_\epsilon^2 \partial_x u_\epsilon - 2x_\epsilon u_\epsilon + u_\epsilon^2 \right] , \qquad u\Big|_{t=0} = u^{\mathrm{in}} ,$$

considered over $[0, 1]$ with the boundary conditions

$$\left[x_\epsilon^2 \partial_x u_\epsilon - 2x_\epsilon u_\epsilon \right] \Big|_{x=0} = 0 ,$$
$$\left[x_\epsilon^2 \partial_x u_\epsilon - 2x_\epsilon u_\epsilon + u_\epsilon^2 \right] \Big|_{x=1} = 0 ,$$

where $x_\epsilon = x + \epsilon$ and $\epsilon > 0$. This problem has been constructed so that it mimics the conservation and entropy structure of our model.

Passing to the limit requires estimates that you will be spared.

Our model problem is easier to analyze than the Kompaneets equation because our regularized problem has the universal super-solution

$$w_\epsilon(x,t) = x_\epsilon + \frac{1-x}{t} + 2t^{-\frac{1}{2}}\,.$$

Every solution u_ϵ of our regularized problem satisfies $u_\epsilon(x,t) < w_\epsilon(x,t)$ over $x \in (0,1]$. Upon passing to the limiting solution u we see that

$$\int_{0^+}^{1} u\,\mathrm{d}x \le \frac{1}{2} + \frac{1}{2t} + 2t^{-\frac{1}{2}}\,.$$

This shows that a Bose-Einstein condensate must develop in finite time for any initial data u^{in} such that

$$\int_{0^+}^{1} u^{\mathrm{in}}\,\mathrm{d}x > \frac{1}{2}\,.$$

Below this threshold a condensate will develop for some initial data, but not for others. For example, no condensate develops if $u^{\mathrm{in}}(x) \le x$.

8. Conclusion

There is much more to do.

- Is there a sharper criterion for a Bose-Einstein condensate to develop?
- Do analogous results hold for the Kompaneets equation?
- Do analogous results hold for other Boson systems? (There have been many recent related results, such as those of Xuguang Lu.)
- Are there numerical schemes that capture this phenomenon?

Thank You!