Quiz 1 Solutions, Math 246, Professor David Levermore
Tuesday, 31 January 2012

(1) [4] For each of the following ordinary differential equations, give its order and state whether it is linear or nonlinear.

(a) $\dfrac{\mathrm{d}^4x}{\mathrm{d}z^4} + z^2\dfrac{\mathrm{d}^2x}{\mathrm{d}z^2} + e^x z = x^2$; **Solution.** fourth order, nonlinear.

(b) $\dfrac{\mathrm{d}^2y}{\mathrm{d}t^2} = 2y\,\dfrac{\mathrm{d}y}{\mathrm{d}t} + \sin(t)$. **Solution.** second order, nonlinear.

(2) [4] Solve the initial-value problem

$$t\frac{\mathrm{d}x}{\mathrm{d}t} = 2x + t^3\,, \qquad x(1) = 0\,.$$

Solution. This equation is linear. Its normal form is

$$\frac{\mathrm{d}x}{\mathrm{d}t} - \frac{2}{t}\,x = t^2\,.$$

An integrating factor is $e^{A(t)}$ where $A'(t) = -2/t$. Setting $A(t) = -2\log(t)$, we find that $e^{A(t)} = e^{-2\log(t)} = t^{-2}$. Hence, the problem has the intgrating factor form

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(t^{-2}x\big) = t^{-2}\cdot t^2 = 1\,.$$

Integrating both sides yields

$$t^{-2}x = t + c\,.$$

Imposing the initial condition gives

$$1^{-2}\cdot 0 = 1 + c\,,$$

whereby $c = -1$. The solution is therefore

$$x = t^3 - t^2\,, \qquad \text{for every } t > 0\,.$$

Remark. The interval of definition for this solution is $(0,\infty)$. Can you see why?

(3) [2] Give the interval of definition for the solution of the initial-value problem

$$\frac{\mathrm{d}z}{\mathrm{d}t} + \frac{1}{t^2-4}\,z = \frac{1}{\sin(t)}\,, \qquad z(-3) = 5\,.$$

(You do not have to solve this equation to answer this question!)

Solution. This equation is linear and is already in normal form. The coefficient $1/(t^2-4)$ is continuous everywhere except at $t = \pm 2$, while the forcing $1/\sin(t)$ is continuous everywhere except where $t = n\pi$ for some integer n. You can therefore read off that the interval of definition for its solution is $(-\pi,-2)$ because:

- the initial time $t = -3$ is in $(-\pi,-2)$,
- the coefficient and forcing are both continuous over $(-\pi,-2)$,
- the coefficient is not defined at $t = -2$,
- the forcing is not defined at $t = -\pi$.