Quiz 4 Solutions, Math 246, Professor David Levermore
Tuesday, 21 February 2012

(1) [3] A tank initially contains 240 liters of pure water. At time $t = 0$ brine (salt water) with a salt concentration of 3 grams per liter (g/l) begins to flow into the tank at a constant rate of 2 liters per minute (l/min) and the well-stirred mixture flows out of the tank at the same rate. Write down an initial-value problem that governs the amount of salt in the tank for $t > 0$. (Do not solve the initial-value problem!)

Solution. Let $S(t)$ denote the mass (g) of salt in the tank at time t minutes. The tank will contain 240 liters of brine for every $t > 0$, so the salt concentration of the brine will be $S(t)/240$ g/l. Because this is also the concentration of the outflow, and because there is no salt in the tank initially, $S(t)$ will satisfy the initial-value problem

$$\frac{\mathrm{d}S}{\mathrm{d}t} = 3 \cdot 2 - \frac{S}{240} \cdot 2 = 6 - \tfrac{1}{120} S \,, \qquad S(0) = 0 \,.$$

(2) [3] In the absence of predators the population of mosquitoes in a certain area would increase at a rate proportional to its current population such that it would triple every two weeks. There are 120,000 mosquitoes in the area when a flock of birds arrives that eats 80,000 mosquitoes per week. Write down an initial-value problem that governs the population of mosquitoes in the area after the flock of birds arrives. (Do not solve the initial-value problem!)

Solution. Let $M(t)$ be the number of mosquitoes at time t weeks. Tripling every two weeks gives a growth factor of $3^{\frac{t}{2}} = (e^{\log(3)})^{\frac{t}{2}} = e^{\frac{1}{2}\log(3)t}$, which implies a growth rate of $\frac{1}{2}\log(3)$. The initial-value problem that M satisfies is therefore

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \tfrac{1}{2}\log(3) M - 80,000 \,, \qquad M(0) = 120,000 \,.$$

(3) [4] Find an implicit general solution of the differential form

$$(6x + 4y)\,\mathrm{d}x + (2y + 4x)\,\mathrm{d}y = 0 \,.$$

Solution. This differential form is exact because

$$\partial_y(6x + 4y) = 4 = \partial_x(2y + 4x) \,.$$

Therefore we can find $H(x, y)$ such that

$$\partial_x H(x, y) = 6x + 4y \,, \qquad \partial_y H(x, y) = 2y + 4x \,.$$

Upon integrating the first equation with respect to x you find

$$H(x, y) = \int 6x + 4y \,\mathrm{d}x = 3x^2 + 4xy + h(y) \,.$$

This implies that $\partial_y H(x, y) = 4x + h'(y)$. By plugging this into the left-hand side of the second equation we obtain $4x + h'(y) = 2y + 4x$, which yields $h'(y) = 2y$. By taking $h(y) = y^2$, we obtain $H(x, y) = 3x^2 + 4xy + y^2$. An implicit general solution is then given by

$$3x^2 + 4xy + y^2 = c \,.$$

Remark. This can be solved for y to obtain the explicit solutions $y = -2x \pm \sqrt{c + x^2}$.