Quiz 5 Solutions, Math 246, Professor David Levermore
Tuesday, 6 March 2012

(1) [2] What is the interval of definition for the solution to the initial-value problem

$$z''' + \frac{3}{t}\, z' + \frac{\cos(5t)}{4+t}\, z = \frac{e^t}{t-3}\,, \qquad z(-2) = z'(-2) = z''(-2) = 6\,.$$

Solution. This linear equation is already in normal form. Both of its coefficients are defined and continuous everywhere except $t = 0$ and $t = -4$. Its forcing is defined and continuous everywhere except $t = 3$. The initial time is $t = -2$. The interval of definition is therefore $(-4, 0)$.

(2) [3] Compute the Wronskian of the functions $Y_1(t) = \cos(5t)$ and $Y_2(t) = \sin(5t)$. (Evaluate the determinant and simplify.)

Solution. Because $Y_1'(t) = -5\sin(5t)$ and $Y_2'(t) = 5\cos(5t)$, the Wronskian is

$$\begin{aligned} W[Y_1, Y_2](t) &= \det\begin{pmatrix} Y_1(t) & Y_2(t) \\ Y_1'(t) & Y_2'(t) \end{pmatrix} = \det\begin{pmatrix} \cos(5t) & \sin(5t) \\ -5\sin(5t) & 5\cos(5t) \end{pmatrix} \\ &= 5\cos(5t)^2 + 5\sin(5t)^2 = 5\,. \end{aligned}$$

(3) [1] Suppose that $Y_1(t)$, $Y_2(t)$, and $Y_3(t)$ are solutions of the differential equation

$$y''' + \cos(t)\, y' - e^t\, y = 0\,,$$

Suppose you know that $W[Y_1, Y_2, Y_3](0) = 7$. What is $W[Y_1, Y_2, Y_3](4)$?

Solution. The equation is in normal form and has coefficients that are continuous everywhere. Because it is third-order while the coefficient of y'' is zero, the Abel Theorem implies that $W[Y_1, Y_2, Y_3](t)$ is constant. We can thereby conclude that $W[Y_1, Y_2, Y_3](4) = W[Y_1, Y_2, Y_3](0) = 7$.

(4) [4] Given that $\cos(5t)$ and $\sin(5t)$ are linearly independent solutions to $y'' + 25y = 0$, find the solution $Y(t)$ to the general initial-value problem

$$y'' + 25y = 0\,, \qquad y(0) = y_0\,, \quad y'(0) = y_1\,.$$

Solution. Let $Y(t) = c_1\cos(5t) + c_2\sin(5t)$. Then $Y'(t) = -5c_1\sin(5t) + 5c_2\cos(5t)$. To satisfy the initial conditions one needs

$$y_0 = Y(0) = c_1\,, \qquad y_1 = Y'(0) = 5c_2\,.$$

It follows that $c_1 = y_0$ and $c_2 = y_1/5$. The solution of the general initial-value problem is therefore

$$Y(t) = y_0\cos(5t) + y_1\frac{\sin(5t)}{5}\,.$$

Remark. The associated natural fundamental set of solutions is thereby

$$N_0(t) = \cos(5t)\,, \qquad N_1(t) = \frac{\sin(5t)}{5}\,.$$