Quiz 7 Solutions, Math 246, Professor David Levermore
Tuesday, 27 March 2012

(1) [3] Give the degree, characteristic, and multiplicity for the forcing term of the equation

$$y'' + 8y' + 25y = 5t^2e^{-4t}\cos(3t)\,.$$

Solution. This is a second-order, nonhomogeneous, linear equation with constant coefficients. Its characteristic polynomial is $p(z) = z^2 + 8z + 25 = (z+4)^2 + 3^2$, which has roots $-4 \pm i3$.

The forcing term $5t^2e^{-4t}\cos(3t)$ has degree $d = 2$, characteristic $\mu + i\nu = -4 + i3$, and multiplicity $m = 1$.

(2) [3] Give a particular solution of the equation

$$y'' + 8y' + 25y = 10e^{-5t}\,.$$

Solution. This is a second-order, nonhomogeneous, linear equation with constant coefficients. Its characteristic polynomial is $p(z) = z^2 + 8z + 25 = (z+4)^2 + 3^2$, which has roots $-4 \pm i3$. Its forcing has degree $d = 0$, characteristic $\mu + i\nu = -5$, and multiplicity $m = 0$.

Undetermined Coefficients. Because $\mu + i\nu = -5$ and $d = m = 0$, there is a particular solution of the form

$$y_P(t) = Ae^{-5t}\,.$$

Because

$$y_P'(t) = -5Ae^{-5t}\,, \qquad y_P''(t) = 25Ae^{-5t}\,,$$

you find that

$$\begin{aligned} y_P'' + 8y_P' + 25y_P &= 25Ae^{-5t} + 8\cdot(-5Ae^{-5t}) + 25Ae^{-5t} \\ &= (25 - 40 + 25)Ae^{-5t} = 10Ae^{-5t} \\ &= 10e^{-5t}\,. \end{aligned}$$

By setting $10A = 10$ you see that a particular solution is $y_P(t) = e^{-5t}$.

Key Identity Evaluation. Because $d + m = 0$ you only need the Key identity

$$\mathrm{L}(e^{zt}) = (z^2 + 8z + 25)e^{zt}\,.$$

By evaluating this at the forcing characteristic $z = -5$ you see that

$$\mathrm{L}(e^{-5t}) = \big((-5)^2 + 8\cdot(-5) + 25\big)e^{-5t} = (25 - 40 + 25)e^{-5t} = 10e^{-5t}\,.$$

You see that a particular solution is $y_P(t) = 2e^{-5t}$.

Remark. To apply the Green method, you would have had to first find that the Green function is given by $g(t) = \frac{1}{3}e^{-4t}\sin(3t)$, and then evaluate the integral

$$y_p(t) = \tfrac{1}{3}\int_0^t e^{-4t+4s}\sin(3t - 3s)\cdot 10e^{-5s}\,\mathrm{d}s\,.$$

This route to the solution takes much longer than the route of either undetermined coefficients or Key Identity evaluations!

(3) [4] A sping-mass system is governed by the initial-value problem

$$h'' + \gamma h' + 9h = 0\,, \qquad h(0) = -1\,, \quad h'(0) = 2\,.$$

(a) Determine the natural frequency and period of the spring.
(b) For what value of γ is the system critically damped?

Solution (a). Because the equation is in normal form, the natural frequency ω_o is

$$\omega_o = \sqrt{9} = 3\,.$$

The natural period is therefore $T_o = 2\pi/\omega_o = \frac{2}{3}\pi$.

Solution (b). The characteristic polynomial is

$$p(z) = z^2 + \gamma z + 9 = (z + \tfrac{1}{2}\gamma)^2 + 9 - \tfrac{1}{4}\gamma^2\,.$$

The system will be critically damped when $9 = \frac{1}{4}\gamma^2$, which is when

$$\gamma = \sqrt{4 \cdot 9} = 2 \cdot 3 = 6\,.$$

Remark. The fact that forcing $5t^2e^{-4t}\cos(3t)$ in problem 1 has degree $d = 2$, characteristic $\mu + i\nu = -4 + i3$, and multiplicity $m = 1$ tells you how to start the methods of undetermined coefficients and Key identity evaluations.

- **Undetermined Coefficients:** It tells you to start by seeking a particular solution in the form

$$\begin{aligned} y_P(t) = {} & (A_0t^3 + A_1t^2 + A_2t)e^{-4t}\cos(2t) \\ & + (B_0t^3 + B_1t^2 + B_2t)e^{-4t}\sin(2t)\,. \end{aligned}$$

  Here the powers of t run from third $(d + m = 3)$ to first $(m = 1)$ while $e^{-4t}\cos(3t)$ and $e^{-4t}\sin(3t)$ are because $\mu + i\nu = -4 + i3$.

- **Key Identity Evaluations:** It tells you to start by evaluating the first $(m = 1)$ through third $(d + m = 3)$ derivatives (with respect to z) of the Key identity at $z = -4 + i3$ $(\mu + i\nu = -4 + i3)$. Because $p(z) = z^2 + 8z + 25$, $p'(z) = 2z + 8$, $p''(z) = 2$, and $p'''(z) = 0$, you see that $p'(-4 + i3) = i6$ and $p''(-4 + i3) = 2$. The evaluations of the first through third derivatives of the Key identity at $z = -4 + i3$ are

$$\begin{aligned} \mathrm{L}\big(t\,e^{-4t+i3t}\big) &= i6 \cdot e^{-4t+i3t}\,, \\ \mathrm{L}\big(t^2e^{-4t+i3t}\big) &= 2 \cdot i6 \cdot t\,e^{-4t+i3t} + 2 \cdot e^{-4t+i3t}\,, \\ \mathrm{L}\big(t^3e^{-4t+i3t}\big) &= 3 \cdot i6 \cdot t^2\,e^{-4t+i3t} + 3 \cdot 2 \cdot t\,e^{-4t+i3t}\,. \end{aligned}$$

Remark. We did not cover the method of Key Identity evaluations in the class lectures. *You will not be tested on it!* However, it is covered in the notes and you are free to use it to solve any problem that also can be solved by the method of undetermined coefficients.