Quiz 8 Solutions, Math 246, Professor David Levermore
Tuesday, 10 April 2012

Short Table: $\mathcal{L}[t^k](s) = \dfrac{k!}{s^{k+1}}$ for $s > 0$, $\mathcal{L}[u(t-c)j(t-c)](s) = e^{-cs}\mathcal{L}[j](s)$.

(1) [3] Use the definition of the Laplace transform to compute $\mathcal{L}[f](s)$ for $f(t) = u(t-5)$, where u is the unit step function.

Solution. By the definitions of the Laplace transform and the unit step function

$$\mathcal{L}[f](s) = \lim_{T\to\infty} \int_0^T e^{-st} f(t)\,dt = \lim_{T\to\infty} \int_0^T e^{-st} u(t-5)\,dt = \lim_{T\to\infty} \int_5^T e^{-st}\,dt .$$

For $s \leq 0$ the above limit diverges because $e^{-st} \geq 1$. For $s > 0$

$$\int_5^T e^{-st}\,dt = -\frac{e^{-st}}{s}\bigg|_5^T = \frac{e^{-s5}}{s} - \frac{e^{-sT}}{s} ,$$

whereby

$$\mathcal{L}[f](s) = \lim_{T\to\infty} \left[\frac{e^{-s5}}{s} - \frac{e^{-sT}}{s} \right] = \frac{e^{-5s}}{s} \quad \text{for } s > 0 .$$

(2) [3] Find the Laplace transform $Y(s)$ of the solution $y(t)$ of the initial-value problem

$y'' + 4y = 0$, $y(0) = 7$, $y'(0) = 0$. DO NOT solve for $y(t)$, just $Y(s)$!

Solution. The Laplace transform of the initial-value problem gives

$$\mathcal{L}[y''](s) + 4\mathcal{L}[y](s) = 0 ,$$

where

$$\mathcal{L}[y](s) = Y(s) ,$$
$$\mathcal{L}[y''](s) = s^2 Y(s) - sy(0) - y'(0) = s^2 Y(s) - 7s .$$

Hence,

$$(s^2 + 4)Y(s) = 7s , \quad \Longrightarrow \quad Y(s) = \frac{7s}{s^2 + 4} .$$

(3) [4] Use the above table to find the Laplace transform $F(s)$ of $f(t) = t - u(t-3)\,t$, where u is the unit step function.

Solution. Because $f(t) = t - u(t-3)j(t-3)$ where $j(t) = t + 3$, item 2 in the table at the top of the page with $c = 3$ gives

$$\begin{aligned} F(s) &= \mathcal{L}[t](s) - \mathcal{L}[u(t-3)j(t-3)](s) \\ &= \mathcal{L}[t](s) - e^{-3s}\mathcal{L}[j](s) = \mathcal{L}[t](s) - e^{-3s}\mathcal{L}[t+3](s) . \end{aligned}$$

Item 1 in the table at the top of the page with $k = 1$ and with $k = 0$ then gives

$$F(s) = \mathcal{L}[t](s) - e^{-3s}\mathcal{L}[t+3](s) = \frac{1}{s^2} - e^{-3s}\left(\frac{1}{s^2} + \frac{3}{s}\right) .$$