Quiz 10 Solutions, Math 246, Professor David Levermore
Tuesday, 24 April 2012

(1) [3] Consider two interconnected tanks filled with brine (salt water). The first tank contains 20 liters and the second contains 16 liters. Well stirred brine flows from the first tank to the second at a rate of 2 liters per hour, and from the second to the first at the same rate. At $t = 0$ there are 30 grams of salt in the first tank and 50 grams in the second. Write down an initial-value problem that governs the amount of salt in each tank as a function of time.

Solution. Let $S_1(t)$ be the grams of salt in the first tank after t hours and $S_2(t)$ be the grams of salt in the second tank. These are governed by the initial-value problem

$$\begin{aligned} \frac{\mathrm{d}S_1}{\mathrm{d}t} &= \frac{S_2}{16}\,2 - \frac{S_1}{20}\,2\,, \qquad & S_1(0) = 30\,, \\ \frac{\mathrm{d}S_2}{\mathrm{d}t} &= \frac{S_1}{20}\,2 - \frac{S_2}{16}\,2\,, \qquad & S_2(0) = 50\,. \end{aligned}$$

(2) [4] Let $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$. Compute $e^{t\mathbf{A}}$.

Solution. The characteristic polynomial is $p(z) = z^2 - 4z + 5 = (z-2)^2 + 1^2$. Hence,

$$\begin{aligned} e^{t\mathbf{A}} &= e^{2t}\big[\cos(t)\mathbf{I} + \sin(t)(\mathbf{A} - 2\mathbf{I})\big] \\ &= e^{2t}\left[\cos(t)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sin(t)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right] \\ &= e^{2t}\begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}\,. \end{aligned}$$

(3) [3] $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}$ has eigenvalues -3 and 2. Find an eigenvector for each eigenvalue.

Solution. One has $\mathbf{A} + 3\mathbf{I} = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}$ and $\mathbf{A} - 2\mathbf{I} = \begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix}$.
The eigenvectors $\mathbf{v}_1$ associated with the eigenvalue -3 satisfy $(\mathbf{A} + 3\mathbf{I})\mathbf{v}_1 = 0$. You can either solve this system or simply read-off from a nonzero column of $\mathbf{A} - 2\mathbf{I}$ that these have the form

$$\mathbf{v}_1 = \alpha_1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} \qquad \text{for some } \alpha_1 \neq 0\,.$$

The eigenvectors $\mathbf{v}_2$ associated with the eigenvalue 2 satisfy $(\mathbf{A} - 2\mathbf{I})\mathbf{v}_2 = 0$. You can either solve this system or simply read-off from a nonzero column of $\mathbf{A} + 3\mathbf{I}$ that these have the form

$$\mathbf{v}_2 = \alpha_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad \text{for some } \alpha_2 \neq 0\,.$$