Modeling Epidemics: Introduction

First Models

- Preliminary goal: Model the spread of an infectious disease through a population.
- Simplifying assumptions:
  - The total population N is constant in time.
  - A newly infected person becomes infectious the next day and remains infectious forever.
  - Each infectious person is equally likely (probability p) to infect each noninfectious person on a given day.
- Let $I(t)$ be the number of infectious people at the start of day t.

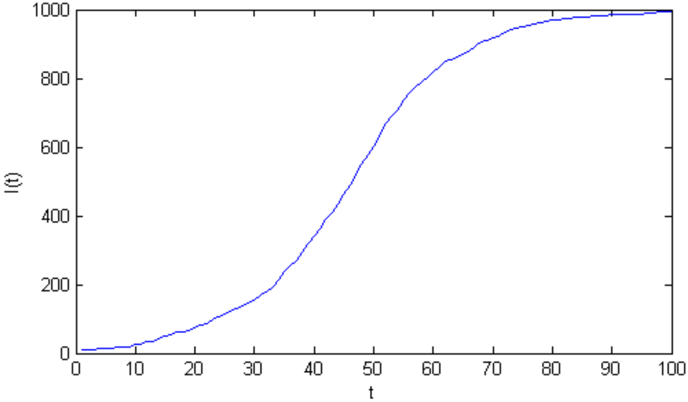
Stochastic Model

- Number the people from 1 to N.
- Let $x_n(t)$ be the infectious status (1 if infectious, 0 if not) of person n at the start of day t.
- We can simulate a possible spread of the disease with the following program ("rand"= random number):

```
for t=1:T-1
    for n=1:N
        let x(n,t+1)=x(n,t)
        for m=1:N
            if x(m,t)=1 and rand<p, then let x(n,t+1)=1
        end
    end
end
```

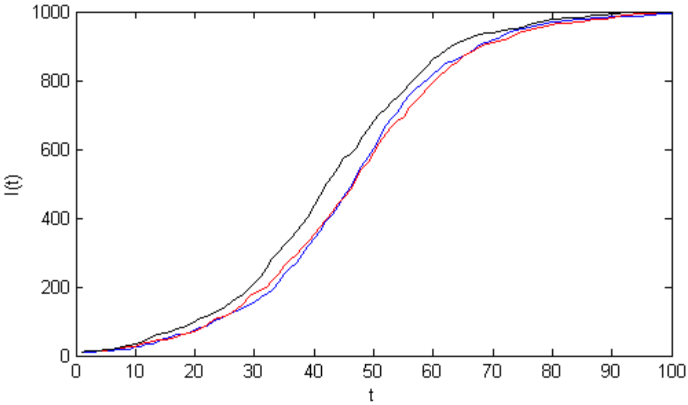
Simulation Results

- Notice that $I(t) = \sum_{n=1}^{N} x_n(t)$.
- Here are the results of a simulation with $p = 10^{-4}$, $N = 1000$, and $I(1) = 10$:



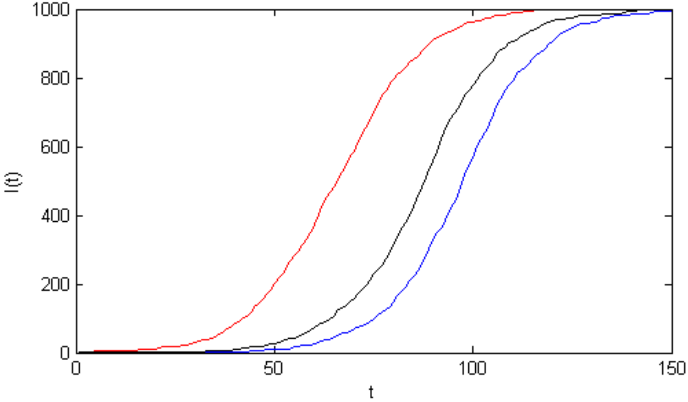
Simulation Results

- And here are the results of three different simulations with $p = 10^{-4}$, $N = 1000$, and $I(1) = 10$:



Simulation Results

- Finally, here are the results of three different simulations with $p = 10^{-4}$, $N = 1000$, and $I(1) = 1$:



Expected (Average) Daily Outcome

- Let's determine the expected number of people infected on a day t that starts with $I(t)$ infectious people and $N - I(t)$ who are susceptible to infection.
- A susceptible person n has probability $1 - p$ of NOT being infected on day t by a given infectious person m. Therefore, person n has probability $(1 - p)^{I(t)}$ of NOT being infected on day t.
- The expected number of people who are infected on day t is then $[1 - (1 - p)^{I(t)}][N - I(t)]$, so

$$E[I(t+1)] = I(t) + [1 - (1 - p)^{I(t)}][N - I(t)]$$

Deterministic Models

- If both $I(t)$ and $N - I(t)$ are large enough, it may be reasonable to approximate $I(t+1)$ by its expected value, resulting in a deterministic model:

$$I(t+1) = I(t) + [1 - (1-p)^{I(t)}][N - I(t)] \qquad (1)$$

- If $pI(t)$ is small, we can approximate $(1-p)^{I(t)}$ by $1 - pI(t)$, yielding a simpler model:

$$I(t+1) = I(t) + pI(t)[N - I(t)] \qquad (2)$$

- For these models, given $I(1)$ we can compute $I(2)$, $I(3)$, ...

Deterministic versus Stochastic

- These deterministic models are much more efficient to compute (1 calculation versus N^2 for the stochastic model). Their predictions may be just as reasonable as any particular simulation of the stochastic model.
- The stochastic model can give some idea of the uncertainty of its predictions via multiple simulations; the deterministic models we've written down say nothing about their uncertainty.

Continuous-Time Model

- The models we have discussed so far are called discrete-time models; time t takes on only integer values.
- We can approximate these models by continuous-time processes; approximating model (2), we get
$$I'(t) = pI(t)[N - I(t)] \tag{3}$$
- We can write down an exact solution to this differential equation:
$$I(t) = \frac{NI(0)}{I(0) + [N - I(0)]e^{-pNt}}$$

Fitting the Model to Data

- The solution $I(t)$ of model (3) has three parameters: N, p, and $I(0)$. Suppose we know N but not the other two parameters. Given a set of data points $[t_j, I_j]$, we can ask which values of p and $I(0)$ best fit the data.
- [A more fundamental (but more difficult) question is whether the model can adequately fit the data at all; are there ANY parameters of the model that fit the data reasonably well?]
- We could try to minimize the sum of the squares of the residuals $I_j - I(t_j)$. However, this would be a NONlinear least squares problem, because $I(t)$ is not a linear function of p or $I(0)$.

Way 1 to use Linear Least Squares

- If the data is given at consecutive values of t, say $t_j = j$, then one approach is to use model (2) and write
$$I(t+1) - I(t) = pI(t)[N - I(t)].$$
The right-hand side is a linear function of the parameter p, and linear least squares yields the value of p that minimizes the sum of the squares of the residuals $I_{j+1} - I_j - pI_j(N - I_j)$.
- This doesn't resolve the question of which value of $I(0)$ to use. If we let $t_0 = 0$ for the first data point, then we could let $I(0) = I_0$. However, this might not be the best choice of $I(0)$ in order to make the residuals $I_j - I(t_j)$ small.

Way 2 to use Linear Least Squares

- Going back to the solution of model (3), we can make a transformation of variables so that the transformed solution does depend linearly on its parameters. First we divide both sides into N and simplify:

$$N/I(t) = 1 + [N/I(0) - 1]e^{-pNt}$$

- Next subtract 1 and take the logarithm:

$$\log[N/I(t) - 1] = \log[N/I(0) - 1] - pNt$$

- Let $Z(t) = \log[N/I(t) - 1]$; then the model becomes $Z(t) = Z(0) - pNt$. This is a linear function of the parameters pN and $Z(0)$. One can transform the data to pairs (t_j, Z_j), use linear least squares to determine values for pN and $Z(0)$, and then solve for p and $I(0)$.

Caveat

- Both ways of using linear least squares transform the model or its solution into a linear relationship between two quantities that can be computed from the data points (t_j, I_j); in the second way, the model predicts that Z_j is a linear function of t_j.
- Rather than simply accept the result of the least squares fit, one should graph the predicted relationship (e.g., Z_j versus t_j) and see if it actually looks linear. This gives some idea of how valid the model is.
- Regardless of how one determines values for p and $I(0)$, one should also check directly how well the resulting $I(t)$ fits the data.