

Modeling Portfolios that Contain Risky Assets

Portfolio Models II: Long Portfolios

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Modeling Portfolios that Contain Risky Assets

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Portfolio Models II: Long Portfolios

Long Portfolio Constraint. *Because the value of any portfolio with short positions has the potential to go negative, many investors do not want to hold a short position in any risky asset.* For these investors we restrict the optimization problem to so-called *long portfolios* by imposing the constraints $\mathbf{f} \geq 0$. Because these are inequality constraints, the resulting optimization problem is not solved by Lagrange multipliers. Here we treat it for cases when $\mathbf{f}_f(\mu_0) \geq 0$ for some μ_0 . The frontier portfolio distribution $\mathbf{f}_f(\mu)$ can be expressed as

$$\mathbf{f}_f(\mu) = \mathbf{f}_f(\mu_0) + \frac{\mu - \mu_0}{\nu_{aS}^2} \mathbf{V}^{-1}(\mathbf{m} - \mu_{mV}\mathbf{1}).$$

Because $\mathbf{1}^T \mathbf{V}^{-1}(\mathbf{m} - \mu_{mV}\mathbf{1}) = b - \mu_{mV}a = 0$, and because $\mathbf{1}$ and \mathbf{m} are not co-linear, we see that the second term above has both positive and negative entries whenever $\mu \neq \mu_0$. Because $\mathbf{f}_f(\mu_0) \geq 0$, the set of μ for which $\mathbf{f}_f(\mu) \geq 0$ is satisfied must be a closed interval containing μ_0 .

Remark. It is often true that $f_{mV} \geq 0$ because the least risky position in a healthy market generally does not require any assets to be held short. In that case it is natural to take $\mu_0 = \mu_{mV}$. However, it is false that $f_{mV} \geq 0$ for every positive definite V . Indeed, for the case $N = 2$ one has

$$V = \begin{pmatrix} v_{11} & v_{12} \\ v_{12} & v_{22} \end{pmatrix}, \quad V^{-1} = \frac{1}{v_{11}v_{22} - v_{12}^2} \begin{pmatrix} v_{22} & -v_{12} \\ -v_{12} & v_{11} \end{pmatrix},$$

while f_{mV} is a positive multiple of

$$V^{-1}\mathbf{1} = \frac{1}{v_{11}v_{22} - v_{12}^2} \begin{pmatrix} v_{22} - v_{12} \\ v_{11} - v_{12} \end{pmatrix}.$$

On one hand, V is positive definite if and only if $v_{11} > 0$, $v_{22} > 0$, and $v_{11}v_{22} > v_{12}^2$. On the other hand, if V is positive definite then $V^{-1}\mathbf{1}$ has nonnegative entries if and only if $v_{11} \geq v_{12}$ and $v_{22} \geq v_{12}$. You can find positive definite matrices for which these conditions do not hold.

Long Frontier. The set of points in the $\sigma\mu$ -plane that correspond to long portfolios that have less volatility than every other long portfolio with the same return rate mean is called the *long frontier*. Because the distribution f of any long portfolio satisfies $\mathbf{1}^T f = 1$ and $f \geq 0$ while its return rate mean is given by $\mu = \mathbf{m}^T f$, we see that μ must satisfy the bounds

$$\mu_{\min} \leq \mu \leq \mu_{\max},$$

where

$$\begin{aligned}\mu_{\min} &= \min\{m_i : i = 1, \dots, N\}, \\ \mu_{\max} &= \max\{m_i : i = 1, \dots, N\}.\end{aligned}$$

It should also be clear to you that every point in $[\mu_{\min}, \mu_{\max}]$ is the return rate mean for some long portfolio. The long frontier is therefore given by $\sigma = \sigma_{lf}(\mu)$ where the function $\sigma_{lf}(\mu)$ is only defined over $[\mu_{\min}, \mu_{\max}]$. As we will see, this function is given by a finite list of formulas that are straightforward to obtain when N is not too large.

Because $\mathbf{f}_f(\mu_0) \geq \mathbf{0}$, the distributions of those frontier portfolios that are long portfolios are given by $\mathbf{f}_f(\mu)$ where $\mu \in [\underline{\mu}_1, \bar{\mu}_1]$ with

$$\begin{aligned}\bar{\mu}_1 &= \max \left\{ \mu \in \mathbb{R} : \mathbf{f}_f(\mu) \geq \mathbf{0} \text{ entrywise} \right\}, \\ \underline{\mu}_1 &= \min \left\{ \mu \in \mathbb{R} : \mathbf{f}_f(\mu) \geq \mathbf{0} \text{ entrywise} \right\}.\end{aligned}$$

The long frontier coincides with the frontier for $\mu \in [\underline{\mu}_1, \bar{\mu}_1]$. That is to say,

$$\sigma_{|f}(\mu) = \sigma_f(\mu) \quad \text{for } \mu \in [\underline{\mu}_1, \bar{\mu}_1].$$

We can extend $\sigma_{|f}(\mu)$ to the interval $[\mu_{mn}, \mu_{mx}]$ by an iterative process. We will show how to do this for the right endpoint. The steps are analogous for the left endpoint. We initialize the iteration by setting

$$\bar{\mathbf{m}}_0 = \mathbf{m}, \quad \bar{\mathbf{V}}_0 = \mathbf{V}, \quad \sigma_{\bar{f}_0}(\mu) = \sigma_f(\mu), \quad \mathbf{f}_{\bar{f}_0}(\mu) = \mathbf{f}_f(\mu).$$

Suppose we have extended $\sigma_{|f}(\mu)$ to an interval with right endpoint $\bar{\mu}_k$. If $\bar{\mu}_k = \mu_{\max}$ then we are done. Otherwise, let the vector $\bar{\mathbf{m}}_k$ and matrix $\bar{\mathbf{V}}_k$ be obtained from $\bar{\mathbf{m}}_{k-1}$ and $\bar{\mathbf{V}}_{k-1}$ by removing every entry with an index corresponding to an entry of $\mathbf{f}_{\bar{f}_{k-1}}(\bar{\mu}_k)$ that is zero. In other words, let $\bar{\mathbf{m}}_k$ be the return rate mean vector and $\bar{\mathbf{V}}_k$ be the return rate covariance matrix after we drop from consideration every asset corresponding to an entry of $\mathbf{f}_{\bar{f}_{k-1}}(\bar{\mu}_k)$ that is zero. (Typically only one asset will be dropped each time.)

Let $\sigma = \sigma_{\bar{f}_k}(\mu)$ be the frontier of this reduced portfolio. The dimension of the associated frontier distribution $\mathbf{f}_{\bar{f}_k}(\mu)$ is less than that of $\mathbf{f}_{\bar{f}_{k-1}}(\mu)$ by the number of zero entries of $\mathbf{f}_{\bar{f}_{k-1}}(\bar{\mu}_k)$. The entries of $\mathbf{f}_{\bar{f}_k}(\bar{\mu}_k)$ are exactly the positive entries of $\mathbf{f}_{\bar{f}_{k-1}}(\bar{\mu}_k)$. Therefore $\sigma_{\bar{f}_k}(\mu)$ satisfies

$$\begin{aligned}\sigma_{\bar{f}_k}(\bar{\mu}_k) &= \mathbf{f}_{\bar{f}_k}(\bar{\mu}_k)^\top \bar{\mathbf{V}}_k \mathbf{f}_{\bar{f}_k}(\bar{\mu}_k) \\ &= \mathbf{f}_{\bar{f}_{k-1}}(\bar{\mu}_k)^\top \bar{\mathbf{V}}_{k-1} \mathbf{f}_{\bar{f}_{k-1}}(\bar{\mu}_k) = \sigma_{\bar{f}_{k-1}}(\bar{\mu}_k).\end{aligned}$$

Because $\sigma_{\bar{f}_k}(\mu)$ is associated with fewer assets, we also know that

$$\sigma_{\bar{f}_k}(\mu) \geq \sigma_{\bar{f}_{k-1}}(\mu) \quad \text{for every } \mu.$$

Because these functions are equal at $\mu = \bar{\mu}_k$, we conclude that moreover

$$\sigma'_{\bar{f}_k}(\bar{\mu}_k) = \sigma'_{\bar{f}_{k-1}}(\bar{\mu}_k).$$

Now let

$$\bar{\mu}_{k+1} = \max\{\mu \in \mathbb{R} : \mathbf{f}_{\bar{f}_k}(\mu) \geq \mathbf{0} \text{ entrywise}\}.$$

Because $\mathbf{f}_{\bar{f}_k}(\bar{\mu}_k) > \mathbf{0}$, it is clear that $\bar{\mu}_{k+1} > \bar{\mu}_k$. Finally, set

$$\sigma_{|f}(\mu) = \sigma_{\bar{f}_k}(\mu) \quad \text{for } \mu \in [\bar{\mu}_k, \bar{\mu}_{k+1}].$$

We have thereby extended $\sigma_{|f}(\mu)$ to an interval with right endpoint $\bar{\mu}_{k+1}$, whereby we can return to the beginning of the iteration.

After applying the analogous iterative process to extend the left endpoint, you find that $\sigma_{|f}(\mu)$ is expressed over $[\mu_{\min}, \mu_{\max}]$ as the list function

$$\sigma_{|f}(\mu) = \begin{cases} \sigma_{\underline{f}_k}(\mu) & \text{for } \mu \in [\underline{\mu}_{k+1}, \underline{\mu}_k], \\ \sigma_f(\mu) & \text{for } \mu \in [\underline{\mu}_1, \bar{\mu}_1], \\ \sigma_{\bar{f}_k}(\mu) & \text{for } \mu \in [\bar{\mu}_k, \bar{\mu}_{k+1}]. \end{cases}$$

This is strictly convex and continuously differentiable over $[\mu_{\min}, \mu_{\max}]$. Its second derivative will have a jump discontinuity at each $\underline{\mu}_k$ and $\bar{\mu}_k$ that lies in (μ_{\min}, μ_{\max}) .

Remark. Here we will not show why the above algorithm for computing $\sigma_{|f}(\mu)$ works. The proof is far more complicated than others in this course. The algorithm is straightforward to implement when N is not too large. When either N is large or no μ_0 exists then $\sigma_{|f}(\mu)$ can be approximated numerically using a *primal-dual interior algorithm for convex optimization*. Such algorithms are taught in some graduate courses on optimization.

Long Frontier Portfolios. Associated with each of the distributions $\underline{f}_{\underline{k}}(\mu)$ and $\bar{f}_{\bar{k}}(\mu)$ of the reduced portfolios in the above construction we define the distributions $\underline{f}_{\underline{k}}(\mu)$ and $\bar{f}_{\bar{k}}(\mu)$ to be the N -vectors obtained by adding zero entries corresponding to assets that are not held by the respective reduced portfolios.

The distributions associated with the long frontier portfolios are then given over $[\mu_{mn}, \mu_{mx}]$ by the list function

$$f_{lf}(\mu) = \begin{cases} \underline{f}_{\underline{k}}(\mu) & \text{for } \mu \in [\underline{\mu}_{k+1}, \underline{\mu}_k], \\ f_f(\mu) & \text{for } \mu \in [\underline{\mu}_1, \bar{\mu}_1], \\ \bar{f}_{\bar{k}}(\mu) & \text{for } \mu \in [\bar{\mu}_k, \bar{\mu}_{k+1}], \end{cases}$$

This is continuous and piecewise linear over $[\mu_{mn}, \mu_{mx}]$. Its first derivative will have a jump discontinuity at each $\underline{\mu}_k$ and $\bar{\mu}_k$ that lies in (μ_{mn}, μ_{mx}) .

Because $f_{lf}(\mu)$ is continuous and piecewise linear over $[\mu_{mn}, \mu_{mx}]$ with nodes $\underline{\mu}_k$ and $\bar{\mu}_k$ in $[\mu_{mn}, \mu_{mx}]$, it can be expressed in terms of the so-called *nodal portfolio distributions* given by

$$\underline{f}_k = \underline{f}_{\underline{f}_k}(\underline{\mu}_k), \quad \bar{f}_k = \bar{f}_{\bar{f}_k}(\bar{\mu}_k).$$

Because

$$\underline{f}_{k+1} = \underline{f}_{\underline{f}_k}(\underline{\mu}_{k+1}), \quad \bar{f}_{k+1} = \bar{f}_{\bar{f}_k}(\bar{\mu}_{k+1}),$$

by the two mutual fund property we have

$$\begin{aligned} \underline{f}_{\underline{f}_k}(\mu) &= \frac{\underline{\mu}_{k+1} - \mu}{\underline{\mu}_{k+1} - \underline{\mu}_k} \underline{f}_k + \frac{\mu - \underline{\mu}_k}{\underline{\mu}_{k+1} - \underline{\mu}_k} \underline{f}_{k+1}, \\ f_f(\mu) &= \frac{\bar{\mu}_1 - \mu}{\bar{\mu}_1 - \underline{\mu}_1} \underline{f}_1 + \frac{\mu - \underline{\mu}_1}{\bar{\mu}_1 - \underline{\mu}_1} \bar{f}_1, \\ \bar{f}_{\bar{f}_k}(\mu) &= \frac{\bar{\mu}_{k+1} - \mu}{\bar{\mu}_{k+1} - \bar{\mu}_k} \bar{f}_k + \frac{\mu - \bar{\mu}_k}{\bar{\mu}_{k+1} - \bar{\mu}_k} \bar{f}_{k+1}. \end{aligned}$$

General Portfolio with Two Risky Assets. Recall the portfolio of two risky assets with mean vector \mathbf{m} and covariance matrix \mathbf{V} given by

$$\mathbf{m} = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}, \quad \mathbf{V} = \begin{pmatrix} v_{11} & v_{12} \\ v_{12} & v_{22} \end{pmatrix}.$$

Here we will assume that $m_1 < m_2$, so that $\mu_{\min} = m_1$ and $\mu_{\max} = m_2$. The frontier portfolios are

$$\mathbf{f}_f(\mu) = \frac{1}{m_2 - m_1} \begin{pmatrix} m_2 - \mu \\ \mu - m_1 \end{pmatrix} \quad \text{for } \mu \in \mathbb{R}.$$

Clearly $\mathbf{f}_f(\mu) \geq \mathbf{0}$ if and only if $\mu \in [m_1, m_2] = [\mu_{\min}, \mu_{\max}]$. Therefore

$$\mathbf{f}_{|f}(\mu) = \mathbf{f}_f(\mu) \quad \text{for } \mu \in [m_1, m_2],$$

and the long frontier is determined by

$$\sigma_{|f}(\mu) = \sigma_f(\mu) = \sqrt{\mathbf{f}_f(\mu)^\top \mathbf{V} \mathbf{f}_f(\mu)} \quad \text{for } \mu \in [m_1, m_2].$$

In this case there is no need to construct $\sigma_{|f}(\mu)$ by the foregoing algorithm.

Simple Portfolio with Three Risky Assets. Recall the portfolio of three risky assets with mean vector \mathbf{m} and covariance matrix \mathbf{V} given by

$$\mathbf{m} = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} = \begin{pmatrix} m - d \\ m \\ m + d \end{pmatrix}, \quad \mathbf{V} = s^2 \begin{pmatrix} 1 & r & r \\ r & 1 & r \\ r & r & 1 \end{pmatrix}.$$

Here $m \in \mathbb{R}$, $d, s \in \mathbb{R}_+$, and $r \in (-\frac{1}{2}, 1)$, where the last condition is equivalent to the condition that \mathbf{V} is positive definite given $s > 0$. Its frontier parameters are

$$\sigma_{\text{mv}} = \sqrt{\frac{1}{a}} = s \sqrt{\frac{1 + 2r}{3}}, \quad \mu_{\text{mv}} = \frac{b}{a} = m,$$

$$\nu_{\text{as}} = \sqrt{c - \frac{b^2}{a}} = \frac{d}{s} \sqrt{\frac{2}{1 - r}}.$$

Its minimum volatility portfolio is $\mathbf{f}_{\text{mv}} = \frac{1}{3}\mathbf{1}$, whereby we can take $\mu_0 = m$. Clearly $[\mu_{\text{mn}}, \mu_{\text{mx}}] = [m - d, m + d]$.

Its frontier is determined by

$$\sigma_f(\mu) = s \sqrt{\frac{1+2r}{3} + \frac{1-r}{2} \left(\frac{\mu-m}{d} \right)^2} \quad \text{for } \mu \in (-\infty, \infty),$$

while the distribution of the frontier portfolio with return rate mean μ is

$$\mathbf{f}_f(\mu) = \begin{pmatrix} \frac{1}{3} - \frac{\mu-m}{2d} \\ \frac{1}{3} \\ \frac{1}{3} + \frac{\mu-m}{2d} \end{pmatrix} = \begin{pmatrix} \frac{m+\frac{2}{3}d-\mu}{2d} \\ \frac{1}{3} \\ \frac{\mu-m+\frac{2}{3}d}{2d} \end{pmatrix}.$$

The frontier portfolio holds long positions when $\mu \in (m - \frac{2}{3}d, m + \frac{2}{3}d)$. Therefore $[\underline{\mu}_1, \bar{\mu}_1] = [m - \frac{2}{3}d, m + \frac{2}{3}d]$ and the long frontier satisfies

$$\sigma_{lf}(\mu) = \sigma_f(\mu) \quad \text{for } \mu \in [m - \frac{2}{3}d, m + \frac{2}{3}d].$$

The distribution weight of first asset vanishes at the right endpoint while that of the third vanishes at the left endpoint.

In order to extend the long frontier beyond the right endpoint $\bar{\mu}_1 = m + \frac{2}{3}d$ to $\mu_{\max} = m + d$ we reduce the portfolio by removing the first asset and set

$$\bar{\mathbf{m}}_1 = \begin{pmatrix} m_2 \\ m_3 \end{pmatrix} = \begin{pmatrix} m \\ m + d \end{pmatrix}, \quad \bar{\mathbf{V}}_1 = s^2 \begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix}.$$

Then

$$\bar{\mathbf{V}}_1^{-1} = \frac{1}{s^2(1-r^2)} \begin{pmatrix} 1 & -r \\ -r & 1 \end{pmatrix}, \quad \bar{\mathbf{V}}_1^{-1} \mathbf{1} = \frac{1}{s^2(1+r)} \mathbf{1},$$

whereby

$$\begin{aligned} \bar{a}_1 &= \mathbf{1}^\top \bar{\mathbf{V}}_1^{-1} \mathbf{1} = \frac{2}{s^2(1+r)}, & \bar{b}_1 &= \mathbf{1}^\top \bar{\mathbf{V}}_1^{-1} \bar{\mathbf{m}}_1 = \frac{2m+d}{s^2(1+r)}, \\ \bar{c}_1 &= \bar{\mathbf{m}}_1^\top \bar{\mathbf{V}}_1^{-1} \bar{\mathbf{m}}_1 = \frac{2m(m+d)}{s^2(1+r)} + \frac{d^2}{s^2(1-r^2)}. \end{aligned}$$

The associated frontier parameters are

$$\sigma_{mv_1} = \sqrt{\frac{1}{\bar{a}_1}} = s \sqrt{\frac{1+r}{2}}, \quad \mu_{mv_1} = \frac{\bar{b}_1}{\bar{a}_1} = m + \frac{1}{2}d,$$

$$\nu_{as_1} = \sqrt{\bar{c}_1 - \frac{\bar{b}_1^2}{\bar{a}_1}} = \frac{d}{2s} \sqrt{\frac{2}{1-r}},$$

whereby the frontier of the reduced portfolio is given by

$$\sigma_{\bar{f}_1}(\mu) = s \sqrt{\frac{1+r}{2} + \frac{1-r}{2} \left(\frac{\mu - m - \frac{1}{2}d}{\frac{1}{2}d} \right)^2}.$$

Similarly, to extend beyond the left endpoint we remove the third asset and find that the frontier of the reduced portfolio is given by

$$\sigma_{\underline{f}_1}(\mu) = s \sqrt{\frac{1+r}{2} + \frac{1-r}{2} \left(\frac{\mu - m + \frac{1}{2}d}{\frac{1}{2}d} \right)^2}.$$

By putting these pieces together we see that the long frontier is given by

$$\sigma_{\text{lf}}(\mu) = \begin{cases} s \sqrt{\frac{1+r}{2} + \frac{1-r}{2} \left(\frac{\mu-m+\frac{1}{2}d}{\frac{1}{2}d} \right)^2} & \text{for } \mu \in [m-d, m-\frac{2}{3}d], \\ s \sqrt{\frac{1+2r}{3} + \frac{1-r}{2} \left(\frac{\mu-m}{d} \right)^2} & \text{for } \mu \in [m-\frac{2}{3}d, m+\frac{2}{3}d], \\ s \sqrt{\frac{1+r}{2} + \frac{1-r}{2} \left(\frac{\mu-m-\frac{1}{2}d}{\frac{1}{2}d} \right)^2} & \text{for } \mu \in [m+\frac{2}{3}d, m+d]. \end{cases}$$

This is strictly convex and continuously differentiable over $[m-d, m+d]$. Its second derivative is defined and positive everywhere in $[m-d, m+d]$ except at the points $\mu = m \pm \frac{2}{3}d$ where it has jump discontinuities. We have

$$\sigma_{\text{lf}}(m \pm \frac{2}{3}d) = s \sqrt{\frac{5+4r}{9}}, \quad \sigma_{\text{lf}}(m \pm d) = s.$$

Finally, the long frontier distributions are given by

$$f_{lf}(\mu) = \begin{cases} \begin{pmatrix} \frac{m-\mu}{d} \\ \frac{\mu-m+d}{d} \\ 0 \end{pmatrix} & \text{for } \mu \in [m-d, m - \frac{2}{3}d], \\ \begin{pmatrix} \frac{1}{3} - \frac{\mu-m}{2d} \\ \frac{1}{3} \\ \frac{1}{3} + \frac{\mu-m}{2d} \end{pmatrix} & \text{for } \mu \in [m - \frac{2}{3}d, m + \frac{2}{3}d], \\ \begin{pmatrix} 0 \\ \frac{m+d-\mu}{d} \\ \frac{\mu-m}{d} \end{pmatrix} & \text{for } \mu \in [m + \frac{2}{3}d, m + d]. \end{cases}$$

Notice that the distribution weights do not depend on either s or r . They are continuous and piecewise linear over $[m-d, m+d]$. Their first derivatives are defined everywhere in $[m-d, m+d]$ except at the points $\mu = m \pm \frac{2}{3}d$ where they have jump discontinuities.

Exercise. Find a 2×2 positive definite matrix V such that the vector $V^{-1}\mathbf{1}$ has a negative entry.

Exercise. Consider the following groups of assets:

- (a) Google, Microsoft, Exxon-Mobil, UPS, GE, and Ford stock in 2009;
- (b) Google, Microsoft, Exxon-Mobil, UPS, GE, and Ford stock in 2007;
- (c) S&P 500 and Russell 1000 and 2000 index funds in 2009;
- (d) S&P 500 and Russell 1000 and 2000 index funds in 2007.

For group (a), group (c), and groups (a) and (c) combined, determine if $f_f(\mu_0) \geq 0$ for some μ_0 . If so, add plots of the associated long frontiers to the graph you produced for these assets in the last exercise of the last section. (Use daily data.) Do the same thing for groups (b) and (d). Explain any relationships you see between the objects plotted on each graph. For which of these groupings is $f_{mv} \geq 0$? Compute $f_{lf}(\mu)$ for each of these groupings, identifying the nodal portfolios.