Navier-Stokes Equations With Supercritical Initial Data

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Navier-Stokes Equations

The Navier-Stokes equations are given by

$$\begin{aligned} & v_t + v \cdot \nabla v - \mu \Delta v + \nabla p = 0, \\ & \nabla \cdot v = 0, \end{aligned}$$

where v is the velocity field, p is the pressure, and $\mu > 0$ is the viscosity coefficient which for simplicity we set $\mu = 1$.

It is well-known that there exists a global weak solution with initial data in L^2.

However, *uniqueness and regularity of weak solution are still open*.

This talk is NOT about

1. Non-uniqueness of weak solution on a hyperbolic space
2. Size of singular set of weak solution.

Leray weak solution of NS

For $v_0 \in L^2$ with $\nabla \cdot v_0 = 0$, there exists a solution v of the Navier-Stokes equations in the sense of distributions: for any smooth divergence-free $\varphi \in \mathscr{C}_c^\infty$,

$$\int\int [v \cdot \varphi_t + v \cdot \Delta\varphi + (v \otimes v) : \nabla\varphi] \, dxdt = 0$$

holds. Moreover, v satisfies the following energy inequality:

$$\|v(t)\|_{L^2}^2 + \int_0^t \|\nabla v(s)\|_{L^2}^2 ds \le \|v_0\|_{L^2}^2.$$

For **Regularity**, **Serrin** proved that the Leray weak solution v is smooth for $t \in (0, T]$ if

$$v \in L^r(0, T; L^s), \quad \frac{2}{r} + \frac{3}{s} = 1$$

The pair (r, s) satisfying the Serrin condition is related the **scaling invariance** property of the Navier-Stokes equations.

Assume that (v, p) solves NS. Then, the same is true for rescaled functions:

$$v_\lambda(t,x) = \lambda v\left(\lambda^2 t, \lambda x\right), \quad p_\lambda(t,x) = \lambda^2 p\left(\lambda^2 t, \lambda x\right).$$

1. $s > \dfrac{d}{2} - 1$ (**Subcritical**)
   (i) Local-in-time solution for large data in H^s

   (ii) Energy method, Sobolev embedding

2. $s = \dfrac{d}{2} - 1$ (**Critical**)
   (i) Global-in-time solution for small data in $\dot{H}^s$

   (ii) Energy method, Sobolev embedding, **Critical norm** (Serrin Condition)

3. $s < \dfrac{d}{2} - 1$ (**Supercritical**): No ill-posed/well-posed results in the setting of **Mild Solution**.

There are some **ill-posedness** results with supercritical initial data in **Dispersive** equations.

1. N. Burq, P. Gerad, N. Tzvetkov

   (i) An instability property of the nonlinear Schrodinger equation on S^d (2002)

   (ii) Two singular dynamics of the nonlinear Schrodinger equation on a plane domain (2003)

2. M. Christ, J. Colliander, T. Tao: Ill-posedness for nonlinear Schrodinger and wave equations (2003)

3. G. Lebeau: Perte de régularité pour les equations $\acute{d}$ondes sur-critiques (2005)

One of the reason is that even for small data which dictates the equation behaves as a linear equation, the free solution does not provide enough **integrability**. A natural question is that

Can we improve integrability of the linear part while keeping regularity?

A positive answer was provided by **Burq–Tzvetkov** [2008]. They proved

1. the local-wellposedness for the wave equations
2. with **supercritical** initial data in the mild solution setting
3. by using the **randomization method.**
4. **This process increases integrability of the linear term while keeping regularity.**

Although this result is very recent, there are already many results using their method;

1. Dispersive equations: Burq–Tzvetkov (2008), Thomann (2009), Oh (2011), Colliander–Oh (2012)
2. Navier-Stokes equations: Deng–Shangbin (2011), Fang–Zhang (2011)

In this talk,

1. Main idea of Burq–Tzvetkov

2. Navier-Stokes equations with Supercritical initial data

Wave Equations: Burq–Tzvetkov

We consider the wave equation on $\mathbb{T}^d$;

$$u_{tt} - \Delta u + |u|^{p-1}u = 0,$$
$$(u, u_t)\big|_{t=0} = (f_1, f_2) \in H^s(\mathbb{T}^d) \times H^{s-1}(\mathbb{T}^d) := \mathscr{H}^s(\mathbb{T}^d).$$

We consider the case $d = 3$ and $p = 3$. Then, the above equation has the scaling:

$$u_\lambda(t, x) = \lambda u(\lambda t, \lambda x).$$

Therefore, $\dot{H}^{\frac{1}{2}}$ is the scaling invariant space.

By using Strichartz estimates, one can show

1. local existence of a strong solution for the subcritical case, $s > \frac{1}{2}$
2. global existence of a mild solution for the critical case, $s = \frac{1}{2}$.

However, the argument to construct a local solutions by Strichartz estimates breaks down for $s < \frac{1}{2}$.

Setting

1. $\{e_n\}$ is an orthonormal basis of $L^2(\mathbb{T}^3)$ constructed from real eigenvectors of the operator $-\Delta$ associated with eigenvalues λ_n^2:

$$-\Delta e_n = \lambda_n^2 e_n.$$

2. For $f = \sum_{n\geq 0} \alpha_n e_n(x)$, we define the energy norms by

$$\|f\|^2_{H^s(\mathbb{T}^3)} = \|(1-\Delta)^{\frac{s}{2}} f\|^2_{L^2(\mathbb{T}^3)} = \sum_{n\geq 0} |\alpha_n|^2 \left(1+\lambda_n^2\right)^s.$$

3. $\{h_n(\omega)\}$ is a sequence of independent, mean zero, and complex random variables on a probability space $(\Omega, \mathscr{A}, p)$ such that for all $n \geq 0$,

$$\int_\Omega |h_n(\omega)|^{2k} dp(\omega) < C$$

holds for some $k \in \mathbb{N}$.

4. **Randomization:** For $f = \sum_{n \geq 0} \alpha_n e_n(x)$, we define the map $(\Omega, \mathscr{A}) \to H^s(\mathbb{T}^3)$ by

$$\omega \longmapsto f^\omega = \sum_{n \geq 0} \alpha_n \mathbf{h_n}(\omega) e_n(x) \in L^2\left(\Omega; H^s(\mathbb{T}^3)\right).$$

This randomization of initial data does not give a regularization in the scale of the Sobolev spaces.

Theorem [Burq–Tzvetkov (2008)]

Let $f \in \mathscr{H}^{\frac{1}{4}}(\mathbb{T}^3)$, with $f^\omega \in L^2\left(\Omega; \mathscr{H}^{\frac{1}{4}}(\mathbb{T}^3)\right)$. Then, for almost all $\omega \in \Omega$, there exists $T_\omega > 0$ and a unique solution u of the cubic wave equations such that

$$u - \left(\cos\left(t\sqrt{-\Delta}\right) f_1^\omega + \frac{\sin\left(t\sqrt{-\Delta}\right) f_2^\omega}{\sqrt{-\Delta}}\right) \in C\left([-T_\omega, T_\omega]; H^{\frac{1}{2}}(\mathbb{T}^3)\right).$$

Strichartz estimates

We say a pair (q, r) is **wave admissible** if

$$\frac{1}{q} + \frac{n-1}{2r} = \frac{n-1}{4}, \quad n \geq 2.$$

Suppose that (q, r) and $(\tilde{q}, \tilde{r})$ are admissible pairs and u is a solution of

$$u_{tt} - \Delta u = F, \quad (u, u_t)\big|_{t=0} = (f_1, f_2) \in \dot{H}^{\gamma} \times \dot{H}^{\gamma-1}.$$

Then, the following **Strichartz estimation** holds:

$$\|u\|_{L^q([0,T];L^r)} + \|u\|_{C([0,T];\dot{\mathscr{H}}^{\gamma})} \lesssim \|f\|_{\dot{H}^{\gamma}} + \|g\|_{\dot{H}^{\gamma-1}} + \|F\|_{L^{\tilde{q}'}([0,T];L^{\tilde{r}'})},$$

where $\quad \dfrac{1}{q} + \dfrac{n}{r} = \dfrac{n}{2} - \gamma = \dfrac{1}{\tilde{q}'} + \dfrac{n}{\tilde{r}'} - 2.$

In particular, $(4, 4)$ is $s = \frac{1}{2}$ admissible for the cubic wave equations in 3D.

Eigenfunction estimates: Sogge(1988)

$$\|e_n\|_{L^p} \le \begin{cases} \lambda^{\frac{d-1}{2}(\frac{1}{2}-\frac{1}{p})} & \text{for} \quad 2 \le p \le \frac{2(d+1)}{d-1} \\ \lambda^{d(\frac{1}{2}-\frac{1}{p})-\frac{1}{2}} & \text{for} \quad \frac{2(d+1)}{d-1} \le p \le \infty \end{cases}$$

for the spectral projection on $\sqrt{-\Delta} \in [\lambda, \lambda+1]$. In 3D,

$$\|e_n\|_{L^4} \lesssim \lambda_n^{\frac{1}{4}} \qquad \text{Cf: Critical space} \quad \dot{H}^{\frac{1}{2}}.$$

Therefore, we can reduce initial regularity by $\frac{1}{4}$: $f \in \dot{\mathscr{H}}^{\frac{1}{4}}$.

Averaging Lemma: L^2 to L^p

Let $\{h_n(\omega)\}$ be a $L^{2k}(\Omega)$ sequence of independent, mean zero and complex random variables. Then, for $p \in [2, 2k]$ and for every complex valued l^2 sequence $c = \{c_n\}$,

$$\Big\| \sum_{n \ge 1} c_n h_n(\omega) \Big\|_{L^p(\Omega)} \lesssim \|c\|_{l^2}.$$

Estimation of the free solution

$$u_f^\omega = \cos(t\sqrt{-\Delta})f_1^\omega + \frac{\sin(t\sqrt{-\Delta})f_2^\omega}{\sqrt{-\Delta}}.$$

By the Minkowski inequality,

$$\left\| e^{it\sqrt{-\Delta}} f_1^\omega \right\|_{L^4(\Omega; L^4([0,T]\times\mathbb{T}^3))} \leq \left\| e^{it\sqrt{-\Delta}} f_1^\omega \right\|_{L^4([0,T]\times\mathbb{T}^3; \mathbf{L^4(\Omega)})}.$$

By averaging lemma,

$$\begin{aligned}
&\left\| e^{it\sqrt{-\Delta}} f_1^\omega \right\|_{L^4([0,T]\times\mathbb{T}^3; \mathbf{L^4(\Omega)})} \\
&\lesssim \left\| \left(\sum |\alpha_n e_n(x)|^2 \right)^{\frac{1}{2}} \right\|_{L^4([0,T]\times\mathbb{T}^3)} \\
&\leq \left(\sum \left\| |\alpha_n e_n(x)|^2 \right\|_{L^2([0,T]\times\mathbb{T}^3)} \right)^{\frac{1}{2}} = T^{\frac{1}{4}} \left(\sum |\alpha_n|^2 \, \|e_n(x)\|_{L^4(\mathbb{T}^3)}^2 \right)^{\frac{1}{2}} \\
&\lesssim T^{\frac{1}{4}} \left(\sum |\alpha_n|^2 (1+\lambda_n^2)^{\frac{2}{8}} \right)^{\frac{1}{2}} = T^{\frac{1}{4}} \|f_1\|_{H^{\frac{1}{4}}(\mathbb{T}^3)}.
\end{aligned}$$

NS with Supercritical Data

We consider NS with initial data in $\dot{H}^s$ for $s < \frac{1}{2}$. By the energy estimation,

$$\sup_{0 \le t \le T} \|v(t)\|_{\dot{H}^s} + \|\nabla v\|_{L^2(0,T;\dot{H}^s)} \lesssim \|v_0\|_{\dot{H}^s} + \|v^2\|_{L^2(0,T;\dot{H}^s)}.$$

$$\nabla^s v \in L^\infty(0, T; L^2) \cap L^2(0, T; L^6) \Longrightarrow \nabla^s v \in L^4(0, T; L^3).$$

By the product rule,

$$\|v^2\|_{L^2(0,T;\dot{H}^s)} \lesssim \|\nabla^s v\|_{L^4(0,T;L^3)} \|v\|_{L^4(0,T;L^6)}.$$

Therefore, we have

$$\|\nabla^s v\|_{L^4(0,T;L^3)} \lesssim \|v_0\|_{\dot{H}^s} + \|\nabla^s v\|_{L^4(0,T;L^3)} \|\mathbf{v}\|_{\mathbf{L^4(0,T;L^6)}}.$$

We need to estimate v in $L^4(0, T; L^6)$ and the norm $\|v\|_{L^4(0,T;L^6)}$ should be sufficiently small to complete the estimation.

We note that the $L^4(0, T; L^6)$ norm is invariant under the scaling

$$v_\lambda(t,x) = \lambda v\left(\lambda^2 t, \lambda x\right).$$

The scaling invariance can be used to solve the Navier-Stokes equations in the integral setting.

We express a solution v in the integral form:

$$v(t) = e^{t\Delta} v_0 - \int_0^t \left[e^{(t-s)\Delta} \mathbb{P} \nabla \cdot (v \otimes v)(s) \right] ds.$$

Any solution satisfying this integral form is called a **mild solution**, and we can find it by using fixed point argument for the function $v \mapsto F(v)$, where

$$F(v)(t) = e^{t\Delta} v_0 - \int_0^t \left[e^{(t-s)\Delta} \mathbb{P} \nabla \cdot (v \otimes v)(s) \right] ds.$$

We now estimate $\|v\|_{L^4(0,T;L^6)}$ by using this integral equation.

Nonlinear Term

$$\left\| \int_0^t \left[\nabla e^{-(t-s)\Delta} \mathbb{P} v^2(s) \right] ds \right\|_{L^6} \lesssim \int_0^t \left[(t-s)^{-\frac{3}{4}} \|v(s)\|_{L^6}^2 \right] ds.$$

By Hardy-Littlewood-Sobolev inequality,

$$\left\| \int_0^t \left[\nabla e^{-(t-s)\Delta} v^2(s) \right] ds \right\|_{L_t^4 L^6} \lesssim \|v\|_{L^4(0,T;L^6)}^2.$$

Linear Term $e^{t\Delta} v_0$

$$\nabla^s e^{t\Delta} v_0 \in L^\infty(0,T;L^2) \cap L^2(0,T;\dot{H}^1).$$

By the interpolation and the Sobolev embedding,

$$e^{t\Delta} v_0 \in L^4\left(0,T;\dot{H}^{s+\frac{1}{2}}\right) \subset L^4\left(0,T;L^{\frac{3}{1-s}}\right), \quad \frac{3}{1-s} < 6.$$

To use the randomization method, we consider the Navier-Stokes equation on $\mathbb{T}^3$.

Main Idea

1. We need $L_t^4 L^6$ norm which corresponds to $\dot{H}^{\frac{1}{2}}$ initial data.
2. $\|e_n\|_{L^p} \lesssim \lambda^{d(\frac{1}{2}-\frac{1}{p})-\frac{1}{2}}$ for $p \geq 4 \quad \Longrightarrow \quad \mathbf{v_0} \in \mathbf{L^2}$.
3. Randomize initial data
4. Regularizing effect of the Heat Kernel.
5. Mild solution: $\|v_0\|_{L^2}$ should be small to obtain a global-in-time solution.

Theorem: There exists $\epsilon > 0$ such that for $v_0 \in L^2$ with $\|v_0\|_{L^2} \leq \epsilon$, there exists an event Ω_ϵ such that $p(\Omega_\epsilon) \geq \dfrac{1}{2}$ and for every $\omega \in \Omega_\epsilon$ there exists a unique global-in-time solution v such that

$$v - e^{t\Delta} v_0^\omega \in L^4\left(0, \infty; L^6(\mathbb{T}^3)\right).$$

Proof: We represent initial data as Fourier series:

$$v_0 = \sum_{n\geq 0} \alpha_n e_n(x) \Longrightarrow e^{t\Delta} v_0^\omega = \sum_{n\geq 0} e^{-t\lambda_n^2} h_n(\omega)\alpha_n e_n(x).$$

Averaging over $\omega \in \Omega$,

$$\left\| e^{t\Delta} v_0^\omega \right\|_{\mathbf{L}^6(\boldsymbol{\Omega}; L_t^4 L^6)} \lesssim \left\| \left(\sum_{n\geq 0} \alpha_n^2 e^{-t\lambda_n^2} |e_n(x)|^2 \right)^{\frac{1}{2}} \right\|_{L_t^4 L^6}$$

$$\lesssim \left\| \left(\sum_{n\geq 0} \alpha_n^2 \left\| e^{-t\lambda_n^2} \right\|_{L_t^4}^2 \|e_n\|_{L^6}^2 \right) \right\|^{\frac{1}{2}} \lesssim \left\| \left(\sum_{n\geq 0} \alpha_n^2 \lambda_n^{-1} \|e_n\|_{L^6}^2 \right) \right\|^{\frac{1}{2}}$$

$$\lesssim \|\mathbf{v_0}\|_{\mathbf{L}^2}.$$

By Chebyshev inequality,

$$E_{\lambda, v_0} = \left\{ \omega \in \Omega : \left\| e^{t\Delta} v_0^\omega \right\|_{\mathbf{L}^6(\boldsymbol{\Omega}; L_t^4 L^6)} \geq \lambda \right\} \quad \Longrightarrow \quad p\left(E_{\lambda, v_0}\right) \leq C\lambda^{-6} \|v_0\|_{L^2(\mathbb{T}^3)}^6.$$

$$p\left(E_{\lambda, v_0}^c\right) \geq 1 - C\lambda^{-6} \|v_0\|_{L^2(\mathbb{T}^3)}^6.$$

Let $v = e^{t\Delta} v_0^\omega + u$, where u solves

$$u(t) = -\int_0^t \left[e^{(t-s)\Delta} \mathbb{P} \nabla \cdot \Big((e^{t\Delta} v_0^\omega + u) \otimes (e^{t\Delta} v_0^\omega + u) \Big)(s) \right] ds.$$

We define the map

$$K^\omega : u \longmapsto -\int_0^t \left[e^{(t-s)\Delta} \mathbb{P} \nabla \cdot \Big((e^{t\Delta} v_0^\omega + u) \otimes (e^{t\Delta} v_0^\omega + u) \Big)(s) \right] ds.$$

For $\omega \in E^c_{\lambda, v_0}$,

$$\|u\|_{L^4_t L^6} \lesssim \lambda^2 + \|u\|^2_{L^4_t L^6}.$$

We take $\lambda \lesssim 1$. Then, K^ω is contractive on the ball of radius 1 of $L^4_t L^6$.

By taking initial data v_0 such that $\|v_0\|_{L^2} \lesssim \lambda$,

$$p\left(E^c_{\lambda, v_0} \right) \geq 1 - C\lambda^{-6} \|v_0\|^6_{L^2(\mathbb{T}^3)} \geq \frac{1}{2}.$$

Concluding remarks

1. We show that there exists a unique global-in-time L^2 solution with a large probability if $\|v_0\|_{L^2}$ is sufficiently small.
2. We can show that there exists a local-in-time solution for large L^2 initial data almost surely.
3. By changing the invariant norm, we can show the above two results for all $s \in [0, \frac{1}{2})$.
4. We do not know the global well-posedness for large data in L^2 almost surely.
5. There are no results on the whole spaces.
6. Possible application: 2D Schrodinger equations with quadratic nonlinearity

$$iu_t - \Delta u = u^2.$$

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