Solutions to Final Exam Sample Problems, Math 246, Fall 2012

(1) Consider the differential equation $\dfrac{\mathrm{d}y}{\mathrm{d}t} = (9 - y^2)y^2$.

(a) Identify its equilibrium (stationary) points and classify their stability.
(b) Sketch how solutions move in the interval $-5 \le y \le 5$ (its phase-line portrait).
(c) If $y(0) = -1$, how does the solution $y(t)$ behave as $t \to \infty$?

Solution (a,b). The right-hand side factors as $(3+y)(3-y)y^2$. The stationary solutions are $y = -3$, $y = 0$, and $y = 3$. A sign analysis of $(3+y)(3-y)y^2$ shows that the phase-line portrait for this equation is therefore

$$
\begin{array}{ccccccc}
- & & + & & + & & - \\
\leftarrow\leftarrow\leftarrow\leftarrow & \bullet & \rightarrow\rightarrow\rightarrow\rightarrow & \bullet & \rightarrow\rightarrow\rightarrow\rightarrow & \bullet & \leftarrow\leftarrow\leftarrow\leftarrow \; y \\
 & -3 & & 0 & & 3 & \\
 & \text{unstable} & & \text{semistable} & & \text{stable} &
\end{array}
$$

Solution (c). The phase-line shows that if $y(0) = -1$ then $y(t) \to 0$ as $t \to \infty$.

(2) Solve (possibly implicitly) each of the following initial-value problems. Identify their intervals of definition.

(a) $\dfrac{\mathrm{d}y}{\mathrm{d}t} + \dfrac{2ty}{1+t^2} = t^2\,, \quad y(0) = 1\,.$
(b) $\dfrac{\mathrm{d}y}{\mathrm{d}x} + \dfrac{e^x y + 2x}{2y + e^x} = 0\,, \quad y(0) = 0\,.$

Solution (a). This equation is *linear* and is already in normal form. An integrating factor is

$$\exp\!\left(\int_0^t \frac{2s}{1+s^2}\,\mathrm{d}s\right) = \exp\big(\log(1+t^2)\big) = 1 + t^2\,,$$

so that the integrating factor form is

$$\frac{\mathrm{d}}{\mathrm{d}t}\big((1+t^2)y\big) = (1+t^2)t^2 = t^2 + t^4\,.$$

Integrate this to obtain

$$(1+t^2)y = \tfrac{1}{3}t^3 + \tfrac{1}{5}t^5 + c\,.$$

The initial condition $y(0) = 1$ implies that $c = (1+0^2)\cdot 1 - \frac{1}{3}0^3 - \frac{1}{5}0^5 = 1$. Therefore

$$y = \frac{1 + \frac{1}{3}t^3 + \frac{1}{5}t^5}{1+t^2}\,.$$

This solution exists for every t, so its interval of definition is $(-\infty, \infty)$.

Remark. Because this equation is linear, you can see that the interval of definition of its solution is $(-\infty, \infty)$ without solving it because both its coefficient and forcing are continuous over $(-\infty, \infty)$.

Solution (b). The initial-value problem is

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{e^x y + 2x}{2y + e^x} = 0\,, \qquad y(0) = 0\,.$$

Express this equation in the differential form

$$(e^x y + 2x)\,\mathrm{d}x + (2y + e^x)\,\mathrm{d}y = 0\,.$$

This differential form is *exact* because

$$\partial_y(e^x y + 2x) = e^x \qquad = \qquad \partial_x(2y + e^x) = e^x\,.$$

Therefore we can find $H(x, y)$ such that

$$\partial_x H(x, y) = e^x y + 2x\,, \qquad \partial_y H(x, y) = 2y + e^x\,.$$

The first equation implies $H(x, y) = e^x y + x^2 + h(y)$. Plugging this into the second equation gives

$$e^x + h'(y) = 2y + e^x\,,$$

which yields $h'(y) = 2y$. Taking $h(y) = y^2$, the general solution is

$$e^x y + x^2 + y^2 = c\,.$$

The initial condition $y(0) = 0$ implies that $c = e^0 \cdot 0 + 0^2 + 0^2 = 0$. Therefore

$$y^2 + e^x y + x^2 = 0\,.$$

The quadratic formula then yields the explicit solution

$$y = \frac{-e^x + \sqrt{e^{2x} - 4x^2}}{2}\,.$$

Here the positive square root is taken because that solution satisfies the initial condition. Its interval of definition is the largest interval (x_L, x_R) containing the initial time 0 over which $e^{2x} > 4x^2$. You cannot find the endpoints of this interval explicitly.

(3) Consider the following MATLAB function M-file.

```
function [t,y] = solveit(ti, yi, tf, n)
t = zeros(n + 1, 1); y = zeros(n + 1, 1);
t(1) = ti; y(1) = yi; h = (tf - ti)/n;
for i = 1:n
t(i + 1) = t(i) + h; y(i + 1) = y(i) + h*((t(i))^4 + (y(i))^2);
end
```

Suppose that the input values are ti = 1, yi = 1, tf = 5, and n = 40.

(a) What is the initial-value problem being approximated numerically?
(b) What is the numerical method being used?
(c) What is the timestep (step size)?
(d) What are the output values for the first two approximations, y(2) and y(3)?

Solution (a). The initial-value problem being approximated numerically is

$$\frac{\mathrm{d}y}{\mathrm{d}t} = t^4 + y^2\,, \qquad y(1) = 1\,.$$

Solution (b). The forward Euler (explicit Euler) method is being used.

Solution (c). The time step is

$$\mathrm{h} = \frac{\mathrm{tF} - \mathrm{tI}}{\mathrm{n}} = \frac{5-1}{40} = \frac{4}{40} = \frac{1}{10} = .1\,.$$

Solution (d). By carrying out the "for" loop in the MATLAB code for $i = 1$ and $i = 2$ you obtain the output values

$$\begin{aligned}
\mathrm{t}(2) &= \mathrm{t}(1) + \mathrm{h} = 1 + .1 = 1.1\,,\\
\mathrm{y}(2) &= \mathrm{y}(1) + \mathrm{h}{*}((\mathrm{t}(1))\hat{}4 + (\mathrm{y}(1))\hat{}2) = 1 + .1(1^4 + 1^2) = 1 + .1 \cdot 2 = 1.2\,.\\
\mathrm{t}(3) &= \mathrm{t}(2) + \mathrm{h} = 1.1 + .1 = 1.2\,,\\
\mathrm{y}(3) &= \mathrm{y}(2) + \mathrm{h}{*}((\mathrm{t}(2))\hat{}4 + (\mathrm{y}(2))\hat{}2) = 1.2 + .1\big((1.1)^4 + (1.2)^2\big)\,.
\end{aligned}$$

You DO NOT have to work out the arithmetic to compute y(3)! If you did then you would obtain $y(3) = 1.49041$.

Remark. You should be able to answer similar questions that employ the improved Euler (Runge trapeziodal) method.

(4) Give an explicit real-valued general solution of the following equations.
   (a) $y'' - 2y' + 5y = te^t + \cos(2t)$
   (b) $u'' - 3u' - 10u = te^{-2t}$
   (c) $v'' + 9v = \cos(3t)$

Solution (a). This is a constant coefficient, nonhomogeneous, linear equation. Its characteristic polynomial is

$$p(z) = z^2 - 2z + 5 = (z-1)^2 + 4 = (z-1)^2 + 2^2\,.$$

This has the conjugate pair of roots $1 \pm i2$, which yields a general solution of the associated homogeneous problem

$$y_H(t) = c_1 e^t \cos(2t) + c_2 e^t \sin(2t)\,.$$

A particular solution $y_P(t)$ can be found by either the method of Key Identity Evaluations or the method of Undetermined Coefficients. The characteristics of the forcing terms te^t and $\cos(2t)$ are $r + is = 1$ and $r + is = i2$ respectively. Because these characteristics are different, they should be treated separately.

Key Indentity Evaluations. The forcing term $t\,e^t$ has degree $d = 1$ and characteristic $r + is = 1$, which is a root of $p(z)$ of multiplicity $m = 0$. Because $m = 0$ and $m + d = 1$, we need the Key Identity and its first derivative

$$\begin{aligned}
\mathrm{L}(e^{zt}) &= (z^2 - 2z + 5)e^{zt}\,,\\
\mathrm{L}(t\,e^{zt}) &= (z^2 - 2z + 5)t\,e^{zt} + (2z - 2)\,e^{zt}\,.
\end{aligned}$$

Evaluate these at $z = 1$ to find $\mathrm{L}(e^t) = 4e^t$ and $\mathrm{L}(t\,e^t) = 4t\,e^t$. Dividing the second of these equations by 4 yields $\mathrm{L}(\frac{1}{4}t\,e^t) = t\,e^t$, which implies $y_{P1}(t) = \frac{1}{4}t\,e^t$.

The forcing term $\cos(2t)$ has degree $d = 0$ and characteristic $r + is = i2$, which is a root of $p(z)$ of multiplicity $m = 0$. Because $m = m + d = 0$, we only need the Key Identity,

$$\mathrm{L}(e^{zt}) = (z^2 - 2z + 5)e^{zt}\,.$$

Evaluating this at $z = i2$ to find $\mathrm{L}(e^{i2t}) = (1 - i4)e^{i2t}$ and dividing by $1 - i4$ yeilds

$$\mathrm{L}\left(\frac{e^{i2t}}{1 - i4}\right) = e^{i2t}\,.$$

Because $\cos(2t) = \mathrm{Re}(e^{i2t})$, the above equation implies

$$\begin{aligned} y_{P2}(t) &= \mathrm{Re}\left(\frac{e^{i2t}}{1-i4}\right) = \mathrm{Re}\left(\frac{(1+i4)e^{i2t}}{1^2+4^2}\right) \\ &= \tfrac{1}{17}\,\mathrm{Re}\big((1+i4)e^{i2t}\big) = \tfrac{1}{17}\big(\cos(2t) - 4\sin(2t)\big)\,. \end{aligned}$$

Combining these two particular solutions with the general solution of the associated homogeneous problem found earlier yields the general solution

$$\begin{aligned} y &= y_H(t) + y_{P1}(t) + y_{P2}(t) \\ &= c_1 e^t \cos(2t) + c_2 e^t \sin(2t) + \tfrac{1}{4} t\, e^t + \tfrac{1}{17}\cos(2t) - \tfrac{4}{17}\sin(2t)\,. \end{aligned}$$

Undetermined Coefficients. The forcing term $t\,e^t$ has degree $d = 1$ and characteristic $r + is = 1$, which is a root of $p(z)$ of multiplicity $m = 0$. Because $m = 0$ and $m + d = 1$, we seek a particular solution of the form

$$y_{P1}(t) = A_0 t\, e^t + A_1 e^t\,.$$

Because

$$y'_{P1}(t) = A_0 t\, e^t + (A_0 + A_1)e^t\,, \qquad y''_{P1}(t) = A_0 t\, e^t + (2A_0 + A_1)e^t\,,$$

we see that

$$\begin{aligned} \mathrm{L}y_{P1}(t) &= y''_{P1}(t) - 2y'_{P1}(t) + 5y_{P1}(t) \\ &= \big(A_0 t\, e^t + (2A_0 + A_1)e^t\big) - 2\big(A_0 t\, e^t + (A_0 + A_1)e^t\big) \\ &\quad + 5\big(A_0 t\, e^t + A_1 e^t\big) \\ &= 4A_0 t\, e^t + 4A_1 e^t\,. \end{aligned}$$

Setting $4A_0 t\, e^t + 4A_1 e^t = t\, e^t$, we see that $4A_0 = 1$ and $4A_1 = 0$, whereby $A_0 = \frac{1}{4}$ and $A_1 = 0$. Hence, a particular solution is $y_{P1}(t) = \frac{1}{4} t\, e^t$.

The forcing term $\cos(2t)$ has degree $d = 0$ and characteristic $r + is = i2$, which is a root of $p(z)$ of multiplicity $m = 0$. Because $m = 0$ and $m + d = 0$, we seek a particular solution of the form

$$y_{P2}(t) = A\cos(2t) + B\sin(2t)\,.$$

Because

$$\begin{aligned} y'_{P2}(t) &= -2A\sin(2t) + 2B\cos(2t)\,, \\ y''_{P2}(t) &= -4A\cos(2t) - 4B\sin(2t)\,, \end{aligned}$$

we see that

$$\begin{aligned} \mathrm{L}y_{P2}(t) &= y''_{P2}(t) - 2y'_{P2}(t) + 5y_{P2}(t) \\ &= \big(-4A\cos(2t) - 4B\sin(2t)\big) - 2\big(-2A\sin(2t) + 2B\cos(2t)\big) \\ &\quad + 5\big(A\cos(2t) + B\sin(2t)\big) \\ &= (A - 4B)\cos(2t) + (B + 4A)\sin(2t)\,. \end{aligned}$$

Setting $(A - 4B)\cos(2t) + (B + 4A)\sin(2t) = \cos(2t)$, we see that

$$A - 4B = 1\,, \qquad B + 4A = 0\,.$$

This system can be solved by any method you choose to find $A = \frac{1}{17}$ and $B = -\frac{4}{17}$, whereby a particular solution is

$$y_{P2}(t) = \tfrac{1}{17}\cos(2t) - \tfrac{4}{17}\sin(2t)\,.$$

Combining these two particular solutions with the general solution of the associated homogeneous problem found earlier yields the general solution

$$\begin{aligned} y &= y_H(t) + y_{P1}(t) + y_{P2}(t) \\ &= c_1 e^t \cos(2t) + c_2 e^t \sin(2t) + \tfrac{1}{4} t\, e^t + \tfrac{1}{17}\cos(2t) - \tfrac{4}{17}\sin(2t)\,. \end{aligned}$$

Solution (b). The equation is

$$u'' - 3u' - 10u = te^{-2t}\,.$$

This is a constant coefficient, nonhomogeneous, linear equation. Its characteristic polynomial is

$$p(z) = z^2 - 3z - 10 = (z-5)(z+2)\,.$$

This has the two real roots 5 and -2, which yields a general solution of the associated homogeneous problem

$$u_H(t) = c_1 e^{5t} + c_2 e^{-2t}\,.$$

A particular solution $u_P(t)$ can be found by either the method of Key Identity Evaluations or the method of Undetermined Coefficients.

Key Indentity Evaluations. The forcing term $t\,e^{-2t}$ has degree $d = 1$ and characteristic $r + is = -2$, which is a root of $p(z)$ of multiplicity $m = 1$. Because $m = 1$ and $m + d = 2$, we will need the first and second derivative of the Key Identity, which are computed from the Key Identity as

$$\begin{aligned} \mathrm{L}(e^{zt}) &= (z^2 - 3z - 10)\, e^{zt}\,, \\ \mathrm{L}(t\,e^{zt}) &= (z^2 - 3z - 10)\, t\,e^{zt} + (2z-3)\, e^{zt}\,, \\ \mathrm{L}(t^2 e^{zt}) &= (z^2 - 3z - 10)\, t^2 e^{zt} + 2(2z-3)\, t\,e^{zt} + 2\, e^{zt}\,. \end{aligned}$$

Evaluate the last two of these at $z = -2$ to find

$$\mathrm{L}(t\,e^{-2t}) = -7e^{-2t}\,, \qquad \mathrm{L}(t^2 e^{-2t}) = -14t\,e^{-2t} + 2e^{-2t}\,.$$

By adding $\frac{2}{7}$ of the first to the second we get

$$\mathrm{L}\big(t^2 e^{-2t} + \tfrac{2}{7} t\,e^{-2t}\big) = -14t\,e^{-2t}\,.$$

By dividing this by -14 we obtain

$$\mathrm{L}\big(-\tfrac{1}{14} t^2 e^{-2t} - \tfrac{2}{98} t\,e^{-2t}\big) = t\,e^{-2t}\,,$$

whereby a particular solution is

$$u_P(t) = -\tfrac{1}{14} t^2 e^{-2t} - \tfrac{2}{98} t\,e^{-2t}\,.$$

Therefore a general solution is

$$u(t) = u_H(t) + u_P(t) = c_1 e^{5t} + c_2 e^{-2t} - \tfrac{1}{14} t^2 e^{-2t} - \tfrac{2}{98} t\,e^{-2t}\,.$$

Undetermined Coefficients. The forcing term $t\,e^{-2t}$ has degree $d = 1$ and characteristic $r + is = -2$, which is a root of $p(z)$ of multiplicity $m = 1$. Because $m = 1$ and $m + d = 2$, we seek a particular solution of the form

$$u_P(t) = A_0 t^2 e^{-2t} + A_1 t\, e^{-2t}\,.$$

Because

$$\begin{aligned} u_P'(t) &= -2A_0 t^2 e^{-2t} + (2A_0 - 2A_1)t\, e^{-2t} + A_1 e^{-2t}\,, \\ u_P''(t) &= 4A_0 t^2 e^{-2t} + (-8A_0 + 4A_1)t\, e^{-2t} + (2A_0 - 4A_1)e^{-2t}\,, \end{aligned}$$

we see that

$$\begin{aligned} \mathrm{L}u_P(t) &= u_P''(t) - 3u_P'(t) - 10u_P(t) \\ &= \big(4A_0 t^2 e^{-2t} + (-8A_0 + 4A_1)t\, e^{-2t} + (2A_0 - 4A_1)e^{-2t}\big) \\ &\quad - 3\big(-2A_0 t^2 e^{-2t} + (2A_0 - 2A_1)t\, e^{-2t} + A_1 e^{-2t}\big) \\ &\quad - 10\big(A_0 t^2 e^{-2t} + A_1 t\, e^{-2t}\big) \\ &= -14A_0 t\, e^{-2t} + (2A_0 - 7A_1)e^{-2t}\,. \end{aligned}$$

By setting $-14A_0 t\, e^{-2t} + (2A_0 - 7A_1)e^{-2t} = t\, e^{-2t}$ we see that

$$-14A_0 = 1\,, \qquad 2A_0 - 7A_1 = 0\,.$$

This system can be solved by any method you choose to find $A_0 = -\frac{1}{14}$ and $A_1 = -\frac{2}{98}$, whereby a particular solution is

$$u_P(t) = -\tfrac{1}{14} t^2 e^{-2t} - \tfrac{2}{98} t\, e^{-2t}\,.$$

Therefore a general solution is

$$u(t) = u_H(t) + u_P(t) = c_1 e^{5t} + c_2 e^{-2t} - \tfrac{1}{14} t^2 e^{-2t} - \tfrac{2}{98} t\, e^{-2t}\,.$$

Solution (c). The equation is

$$v'' + 9v = \cos(3t)\,.$$

This is a constant coefficient, nonhomogeneous, linear equation. Its characteristic polynomial is

$$p(z) = z^2 + 9 = z^2 + 3^2\,.$$

This has the conjugate pair of roots $\pm i3$, which yields a general solution of the associated homogeneous problem

$$v_H(t) = c_1 \cos(3t) + c_2 \sin(3t)\,.$$

A particular solution $v_P(t)$ can be found by either the method of Key Identity Evaluations or the method of Undetermined Coefficients.

Key Indentity Evaluations. The forcing term $\cos(3t)$ has degree $d = 0$ and characteristic $r + is = i3$, which is a root of $p(z)$ of multiplicity $m = 1$. Because $m = 1$ and $m + d = 1$, we need the first derivative of the Key Identity, which is found as

$$\begin{aligned} \mathrm{L}(e^{zt}) &= (z^2 + 9)\, e^{zt}\,, \\ \mathrm{L}(t\, e^{zt}) &= (z^2 + 9)\, t\, e^{zt} + 2z\, e^{zt}\,. \end{aligned}$$

Evaluate the first derivative of the Key Identity at $z = i3$ to find that

$$\mathrm{L}(t\, e^{i3t}) = i6\, e^{i3t} \,.$$

Because $\cos(3t) = \mathrm{Re}(e^{i3t})$, upon dividing by $i6$ and taking the real part you see that a particular solution is

$$v_P(t) = \mathrm{Re}\big(\tfrac{1}{i6} t\, e^{i3t}\big) = \mathrm{Re}\big(\tfrac{1}{i6} t\, (\cos(3t) + i\sin(3t))\big) = \tfrac{1}{6} t\, \sin(3t) \,.$$

Therefore a general solution is

$$v(t) = v_H(t) + v_P(t) = c_1 \cos(3t) + c_2 \sin(3t) + \tfrac{1}{6} t\, \sin(3t) \,.$$

Undetermined Coefficients. The forcing term $\cos(3t)$ has degree $d = 0$ and characteristic $r + is = i3$, which is a root of $p(z)$ of multiplicity $m = 1$. Because $m = 1$ and $m + d = 1$, we seek a particular solution of the form

$$v_P(t) = At\, \cos(3t) + Bt\, \sin(3t) \,.$$

Because

$$\begin{aligned}
v_P'(t) &= -3At\, \sin(3t) + 3Bt\, \cos(3t) A\cos(3t) + B\sin(3t) \,, \\
v_P''(t) &= -9At\, \cos(3t) - 9Bt\, \sin(3t) - 6A\sin(3t) + 6B\cos(3t) \,,
\end{aligned}$$

we see that

$$\begin{aligned}
\mathrm{L}v_P(t) &= v_P''(t) + 9v_P(t) \\
&= \big(-9At\, \cos(3t) - 9Bt\, \sin(3t) - 6A\sin(3t) + 6B\cos(3t)\big) \\
&\quad + 9\big(At\, \cos(3t) + Bt\, \sin(3t)\big) \\
&= -6A\sin(3t) + 6B\cos(3t) \,.
\end{aligned}$$

By setting $-6A\sin(3t) + 6B\cos(3t) = \cos(3t)$ we see that $A = 0$ and $B = \frac{1}{6}$, whereby a particular solution is

$$v_P(t) = \tfrac{1}{6} t\, \sin(3t) \,.$$

Therefore a general solution is

$$v(t) = v_H(t) + v_P(t) = c_1 \cos(3t) + c_2 \sin(3t) + \tfrac{1}{6} t\, \sin(3t) \,.$$

Remark. Because of the simple form of this equation, if you had tried to solve it by either the Green Function or Variation of Parameters method then integrals that arise are not too difficult. However, it is not generally a good idea to use these methods for such problems because evaluating the integrals that arise often involve much more work that the methods shown above.

(5) Solve the following initial-value problems.

(a) $w'' + 4w' + 20w = 5e^{2t}, \qquad w(0) = 3 \,, \quad w'(0) = -7.$

(b) $y'' - 4y' + 4y = \dfrac{e^{2t}}{3+t}, \qquad y(0) = 0 \,, \quad y'(0) = 5.$

(c) $tu'' + 4u' = 0, \qquad u(1) = 2 \,, \quad u'(1) = -3.$

You should try to evaluate any definite integrals that arise.

Solution (a). This is a constant coefficient, nonhomogeneous, linear equation. Its characteristic polynomial is

$$p(z) = z^2 + 4z + 20 = (z+2)^2 + 16 = (z+2)^2 + 4^2\,.$$

This has the conjugate pair of roots $-2 \pm i4$, which yields a general solution of the associated homogeneous problem

$$w_H(t) = c_1 e^{-2t}\cos(4t) + c_2 e^{-2t}\sin(4t)\,.$$

A particular solution $w_P(t)$ can be found by either the method of Key Identity Evaluations or the method of Undetermined Coefficients.

Key Indentity Evaluations. The forcing term $5e^{2t}$ has degree $d = 0$ and characteristic $r + is = 2$, which is a root of $p(z)$ of multiplicity $m = 0$. Because $m = 0$ and $m + d = 0$, we only need the Key Identity,

$$\mathrm{L}(e^{zt}) = (z^2 + 4z + 20)\,e^{zt}\,.$$

Evaluate this at $z = 2$ to find that

$$\mathrm{L}(e^{2t}) = (4 + 8 + 20)e^{2t} = 32e^{2t}\,.$$

Upon multiplying this by $\frac{5}{32}$ we see that a particular solution is

$$w_P(t) = \tfrac{5}{32}e^{2t}\,.$$

Undetermined Coefficients. The forcing term $5e^{2t}$ has degree $d = 0$ and characteristic $r + is = 2$, which is a root of $p(z)$ of multiplicity $m = 0$. Therefore we seek a particular solution of the form

$$w_P(t) = Ae^{2t}\,.$$

Because

$$w_P'(t) = 2Ae^{2t}\,, \qquad w_P''(t) = 4Ae^{2t}\,,$$

we see that

$$\begin{aligned}\mathrm{L}w_P(t) &= w_P''(t) + 4w_P'(t) + 20w_P(t)\\ &= 4Ae^{2t} + 4(2Ae^{2t}) + 20Ae^{2t} = 32Ae^{2t}\,.\end{aligned}$$

By setting $32Ae^{2t} = 5e^{2t}$, we see that $A = \frac{5}{32}$, whereby a particular solution is

$$w_P(t) = \tfrac{5}{32}e^{2t}\,.$$

Solving the Initial-Value Problem. By either method we find that a general solution is

$$w(t) = w_H(t) + w_P(t) = c_1 e^{-2t}\cos(4t) + c_2 e^{-2t}\sin(4t) + \tfrac{5}{32}e^{2t}\,.$$

Because

$$\begin{aligned}w'(t) = &-2c_1 e^{-2t}\cos(4t) - 4c_1 e^{-2t}\sin(4t)\\ &- 2c_2 e^{-2t}\sin(4t) + 4c_2 e^{-2t}\cos(4t) + \tfrac{5}{16}e^{2t}\,,\end{aligned}$$

the initial conditions yield

$$3 = w(0) = c_1 + \tfrac{5}{32}\,, \qquad -7 = w'(0) = -2c_1 + 4c_2 + \tfrac{5}{16}\,.$$

Upon solving this system we find that $c_1 = \frac{91}{32}$ and $c_2 = -\frac{13}{32}$, whereby the solution of the initial-value problem is

$$w(t) = \tfrac{91}{32} e^{-2t} \cos(4t) - \tfrac{13}{32} e^{-2t} \sin(4t) + \tfrac{5}{32} e^{2t} \,.$$

Solution (b). The initial-value problem is

$$y'' - 4y' + 4y = \frac{e^{2t}}{3+t} \,, \qquad y(0) = 0 \,, \quad y'(0) = 5 \,.$$

This is a constant coefficient, nonhomogeneous, linear equation in normal form. Its characteristic polynomial is

$$p(z) = z^2 - 4z + 4 = (z-2)^2 \,.$$

This has the double real root 2, which yields a general solution of the associated homogeneous problem

$$y_H(t) = c_1 e^{2t} + c_2 t\, e^{2t} \,.$$

A particular solution $y_P(t)$ *cannot* be found by either the method of Key Identity Evaluations or the method of Undetermined Coefficients. Rather, we must use either the Green Function or the Variation of Parameters method.

Green Function. The associated Green function $g(t)$ satisfies

$$g'' - 4g' + 4g = 0 \,, \qquad g(0) = 0 \,, \quad g'(0) = 1 \,.$$

A general solution of this equation is

$$g(t) = c_1 e^{2t} + c_2 t\, e^{2t} \,.$$

Because $0 = g(0) = c_1$, we see that $g(t) = c_2 t\, e^{2t}$. Then

$$g'(t) = c_2 e^{2t} + 2c_2 t\, e^{2t} \,.$$

Because $1 = g'(0) = c_2$, the Green function is $g(t) = t\, e^{2t}$. The particular solution $y_P(t)$ that satisfies $y_P(0) = y_P'(0) = 0$ is given by

$$\begin{aligned} y_P(t) &= \int_0^t g(t-s) \frac{e^{2s}}{3+s} \, \mathrm{d}s = \int_0^t (t-s) e^{2t-2s} \frac{e^{2s}}{3+s} \, \mathrm{d}s \\ &= e^{2t} \int_0^t \frac{t-s}{3+s} \, \mathrm{d}s = e^{2t} t \int_0^t \frac{1}{3+s} \, \mathrm{d}s - e^{2t} \int_0^t \frac{s}{3+s} \, \mathrm{d}s \,. \end{aligned}$$

Because

$$\begin{aligned} &\int_0^t \frac{1}{3+s} \, \mathrm{d}s = \log(3+s)\Big|_0^t = \log(3+t) - \log(3) = \log\!\left(\frac{3+t}{3}\right), \\ &\int_0^t \frac{s}{3+s} \, \mathrm{d}s = \int_0^t 1 - \frac{3}{3+s} \, \mathrm{d}s = t - 3\log\!\left(\frac{3+t}{3}\right), \end{aligned}$$

we find that

$$y_P(t) = e^{2t} t \log(1 + \tfrac{1}{3}t) - e^{2t}\big(t - 3\log(1 + \tfrac{1}{3}t)\big) \,.$$

Therefore a general solution of the equation is

$$y(t) = c_1 e^{2t} + c_2 t\, e^{2t} + y_P(t) \,.$$

Because

$$y'(t) = 2c_1 e^{2t} + 2c_2 t\, e^{2t} + c_2 e^{2t} + y_P'(t)\,,$$

and because $y_P(0) = y_P'(0) = 0$, the initial conditions imply

$$0 = y(0) = c_1\,, \qquad 5 = y'(0) = 2c_1 + c_2\,.$$

We find that $c_1 = 0$ and $c_2 = 5$, whereby the solution of the initial-value problem is

$$y(t) = 5t\, e^{2t} + e^{2t} t \log(1 + \tfrac{1}{3}t) - e^{2t}\big(t - 3\log(1 + \tfrac{1}{3}t)\big)\,.$$

Variation of Parameters. The equation is already in normal form. We therefore *seek* a particular solution of the form

$$y_P(t) = e^{2t} u_1(t) + t\, e^{2t} u_2(t)\,,$$

such that

$$\begin{aligned} e^{2t} u_1'(t) + t\, e^{2t} u_2'(t) &= 0\,,\\ 2e^{2t} u_1'(t) + (2t\, e^{2t} + e^{2t}) u_2'(t) &= \frac{e^{2t}}{3+t}\,. \end{aligned}$$

This system can be solved to find that

$$u_1'(t) = -\frac{t}{3+t}\,, \qquad u_2'(t) = \frac{1}{3+t}\,.$$

These can be integrted to obtain

$$\begin{aligned} u_1(t) &= -\int \frac{t}{3+t}\,\mathrm{d}t = -\int 1 - \frac{3}{3+t}\,\mathrm{d}t = -t + 3\log(3+t) + c_1\,,\\ u_2(t) &= \int \frac{1}{3+t}\,\mathrm{d}t = \log(3+t) + c_2\,, \end{aligned}$$

whereby a general solution is

$$y(t) = c_1 e^{2t} - e^{2t}\big(t - 3\log(3+t)\big) + c_2 t\, e^{2t} + t\, e^{2t} \log(3+t)\,.$$

Because

$$\begin{aligned} y'(t) = 2c_1 e^{2t} &- 2e^{2t}\big(t - 3\log(3+t)\big) - e^{2t}\left(1 - \frac{3}{3+t}\right)\\ &+ 2c_2 t\, e^{2t} + c_2 e^{2t} + 2t\, e^{2t}\log(3+t) + e^{2t}\log(3+t) + t\, e^{2t}\frac{1}{3+t}\,, \end{aligned}$$

the initial conditions imply that

$$\begin{aligned} 0 &= y(0) = c_1 + 3\log(3)\,,\\ 5 &= y'(0) = 2c_1 + 6\log(3) + c_2 + \log(3)\,. \end{aligned}$$

We can solve this system to find that $c_1 = -3\log(3)$ and $c_2 = 5 - \log(3)$. Therefore the solution of the initial-value problem is

$$y(t) = -3\log(3)e^{2t} - e^{2t}\big(t - 3\log(3+t)\big) + \big(5 - \log(3)\big)t\, e^{2t} + t\, e^{2t}\log(3+t)\,.$$

Solution (c). The initial-value problem is

$$tu'' + 4u' = 0\,, \qquad u(1) = 2\,, \quad u'(1) = -3\,.$$

This is a variable coefficient, homogeneous, second-order linear equation. Because it does not depend explictly on u, we can reduce its order by setting $w = u'$. Then w satisfies the initial-value problem

$$tw' + 4w = 0\,, \qquad w(1) = -3\,.$$

This is a variable coefficient, homogeneous, first-order linear equation. Its normal form is

$$w' + \frac{4}{t}w = 0\,, \qquad w(1) = -3\,.$$

By setting

$$A(t) = \int_1^t \frac{4}{s}\,\mathrm{d}s = 4\log(s)\Big|_1^t = 4\log(t) - 4\log(1) = 4\log(t)\,,$$

the solution of the initial-value problem for w is

$$w(t) = w(1)e^{-A(t)} = -3e^{-4\log(t)} = -3t^{-4}\,.$$

Then because $u' = w$ and $u(1) = 2$ we see that

$$u(t) = u(1) + \int_1^t w(s)\,\mathrm{d}s = 2 - 3\int_1^t s^{-4}\,\mathrm{d}s = 2 + \left[s^{-3}\right]\Big|_1^t = 2 + t^{-3} - 1 = 1 + t^{-3}\,.$$

The interval of definition for this solution is $(0, \infty)$, which can be seen by putting the original initial-value problem into normal form.

(6) Give an explicit general solution of the equation

$$h'' + 2h' + 5h = 0\,.$$

Sketch a typical solution for $t \geq 0$. If this equation governs a damped spring-mass system, is the system over, under, or critically damped?

Solution. This is a constant coefficient, homogeneous, linear equation. Its characteristic polynomial is

$$p(z) = z^2 + 2z + 5 = (z+1)^2 + 2^2\,.$$

This has the conjugate pair of roots $-1 \pm i2$, which yields a general solution

$$h(t) = c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t)\,.$$

When $c_1^2 + c_2^2 > 0$ this can be put into the amplitute-phase form

$$h(t) = Ae^{-t}\cos(2t - \delta)\,,$$

where $A > 0$ and $0 \leq \delta < 2\pi$ are determined from c_1 and c_2 by

$$A = \sqrt{c_1^2 + c_2^2}\,, \qquad \cos(\delta) = \frac{c_1}{A}\,, \qquad \sin(\delta) = \frac{c_2}{A}\,.$$

In other words, (A, δ) are the polar coordinates for the point in the plane whose Cartesian coordinates are (c_1, c_2). The sketch should show a decaying oscillation with amplitude Ae^{-t} and quasiperiod $\frac{2\pi}{2} = \pi$. A sketch might be given during the review session. The equation governs an *under damped* spring-mass system because its characteristic polynomial has a conjugate pair of roots.

(7) When a mass of 2 kilograms is hung vertically from a spring, it stretches the spring 0.5 meters. (Gravitational acceleration is 9.8 m/sec^2.) At $t = 0$ the mass is set in motion from 0.3 meters below its equilibrium (rest) position with a upward velocity of 2 m/sec. It is acted upon by an external force of $2\cos(5t)$. Neglect drag and assume that the spring force is proportional to its displacement. Formulate an initial-value problem that governs the motion of the mass for $t > 0$. (DO NOT solve this initial-value problem; just write it down!)

Solution. Let $h(t)$ be the displacement (in meters) of the mass from its equilibrium (rest) position at time t (in seconds), with *upward displacements being positive.* The governing initial-value problem then has the form

$$m\frac{\mathrm{d}^2h}{\mathrm{d}t^2} + kh = 2\cos(5t)\,, \qquad h(0) = -.3\,, \quad h'(0) = 2\,,$$

where m is the mass and k is the spring constant. The problem says that $m = 2$ kilograms. The spring constant is obtained by balancing the weight of the mass (mg $= 2 \cdot 9.8$ Newtons) with the force applied by the spring when it is stetched .5 m. This gives $k\,.5 = 2 \cdot 9.8$, or

$$k = \frac{2 \cdot 9.8}{.5} = 4 \cdot 9.8 \quad \text{Newtons/m}\,.$$

The governing initial-value problem is therefore

$$2\frac{\mathrm{d}^2h}{\mathrm{d}t^2} + 4 \cdot 9.8h = 2\cos(5t)\,, \qquad h(0) = -.3\,, \quad h'(0) = 2\,.$$

Had you chosen *downward displacements to be positive* then the sign of the initial data would change! You should make your convention clear!

(8) Find the Laplace transform $Y(s)$ of the solution $y(t)$ to the initial-value problem

$$y'' + 4y' + 8y = f(t)\,, \qquad y(0) = 2\,, \quad y'(0) = 4\,.$$

where

$$f(t) = \begin{cases} 4 & \text{for } 0 \le t < 2\,, \\ t^2 & \text{for } 2 \le t\,. \end{cases}$$

You may refer to the Laplace table in the book. (DO NOT take the inverse Laplace transform to find $y(t)$; just solve for $Y(s)$!)

Solution. The Laplace transform of the initial-value problem is

$$\mathcal{L}[y''](s) + 4\mathcal{L}[y'](s) + 8\mathcal{L}[y](s) = \mathcal{L}[f](s)\,,$$

where

$$\begin{aligned} \mathcal{L}[y](s) &= Y(s)\,, \\ \mathcal{L}[y'](s) &= sY(s) - y(0) = sY(s) - 2\,, \\ \mathcal{L}[y''](s) &= s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - 2s - 4\,. \end{aligned}$$

To compute $\mathcal{L}[f](s)$, first write f as

$$\begin{aligned} f(t) &= \big(1 - u(t-2)\big)4 + u(t-2)t^2 = 4 - u(t-2)4 + u(t-2)t^2 \\ &= 4 + u(t-2)(t^2 - 4) = 4 + u(t-2)j(t-2)\,, \end{aligned}$$

where

$$j(t) = (t+2)^2 - 4 = t^2 + 4t\,.$$

Referring to the table of Laplace transforms in the book, item 13 with $c = 2$ and $f(t) = j(t)$, item 1, and item 3 with $n = 1$ and $n = 2$ then show that

$$\begin{aligned}
\mathcal{L}[f](s) &= 4\mathcal{L}[1](s) + \mathcal{L}\big[u(t-2)j(t-2)\big](s) \\
&= 4\mathcal{L}[1](s) + e^{-2s}\mathcal{L}\big[j(t)\big](s) \\
&= 4\mathcal{L}[1](s) + e^{-2s}\mathcal{L}[4t + t^2](s) \\
&= 4\mathcal{L}[1](s) + 4e^{-2s}\mathcal{L}[t](s) + e^{-2s}\mathcal{L}[t^2](s) \\
&= 4\frac{1}{s} + 4e^{-2s}\frac{1}{s^2} + e^{-2s}\frac{2}{s^3}\,.
\end{aligned}$$

The Laplace transform of the initial-value problem then becomes

$$\big(s^2Y(s) - 2s - 4\big) + 4\big(sY(s) - 2\big) + 8Y(s) = \frac{4}{s} + e^{-2s}\frac{4}{s^2} + e^{-2s}\frac{2}{s^3}\,,$$

which becomes

$$(s^2 + 4s + 8)Y(s) - 2s - 12 = \frac{4}{s} + e^{-2s}\frac{4}{s^2} + e^{-2s}\frac{2}{s^3}\,.$$

Hence, $Y(s)$ is given by

$$Y(s) = \frac{1}{s^2 + 4s + 8}\left(2s + 12 + \frac{4}{s} + e^{-2s}\frac{4}{s^2} + e^{-2s}\frac{2}{s^3}\right).$$

(9) Find the function $y(t)$ whose Laplace transform $Y(s)$ is given by

$$\text{(a)}\quad Y(s) = \frac{e^{-3s}4}{s^2 - 6s + 5}\,, \qquad\qquad \text{(b)}\quad Y(s) = \frac{e^{-2s}s}{s^2 + 4s + 8}\,.$$

You may refer to the table in Section 6.2 of the book.

Solution (a). The denominator factors as $(s-5)(s-1)$, so the partial fraction decomposition is

$$\frac{4}{s^2 - 6s + 5} = \frac{4}{(s-5)(s-1)} = \frac{1}{s-5} - \frac{1}{s-1}\,.$$

Referring to the table of Laplace transforms in the book, item 11 with $n = 0$ and $a = 5$, and with $n = 0$ and $a = 1$ gives

$$\mathcal{L}^{-1}\left[\frac{1}{s-5}\right](t) = e^{5t}\,, \qquad \mathcal{L}^{-1}\left[\frac{1}{s-1}\right](t) = e^{t}\,,$$

whereby

$$\mathcal{L}^{-1}\left[\frac{4}{s^2 - 6s + 5}\right](t) = \mathcal{L}^{-1}\left[\frac{1}{s-5} - \frac{1}{s-1}\right] = e^{5t} - e^{t}\,.$$

It follows from item 13 with $c = 3$ and $f(t) = e^{5t} - e^{t}$ that

$$\begin{aligned}
y(t) = \mathcal{L}^{-1}[Y(s)](t) &= \mathcal{L}^{-1}\left[\frac{e^{-3s}4}{s^2 - 6s + 5}\right](t) \\
&= u(t-3)\mathcal{L}^{-1}\left[\frac{4}{s^2 - 6s + 5}\right](t-3) = u(t-3)\big(e^{5(t-3)} - e^{t-3}\big)\,.
\end{aligned}$$

Solution (b). The denominator does not have real factors. The partial fraction decomposition is

$$\frac{s}{s^2+4s+8} = \frac{s}{(s+2)^2+4} = \frac{s+2}{(s+2)^2+2^2} - \frac{2}{(s+2)^2+2^2}\,.$$

Referring to the table of Laplace transforms in the book, items 10 and 9 with $a=-2$ and $b=2$ give

$$\mathcal{L}^{-1}\left[\frac{s+2}{(s+2)^2+2^2}\right](t) = e^{-2t}\cos(2t)\,, \quad \mathcal{L}^{-1}\left[\frac{2}{(s+2)^2+2^2}\right](t) = e^{-2t}\sin(2t)\,,$$

whereby

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{s}{s^2+4s+8}\right](t) &= \mathcal{L}^{-1}\left[\frac{s+2}{(s+2)^2+2^2}\right](t) - \mathcal{L}^{-1}\left[\frac{2}{(s+2)^2+2^2}\right](t)\\ &= e^{-2t}\big(\cos(2t)-\sin(2t)\big)\,.\end{aligned}$$

It follows from item 13 with $c=2$ and $f(t)=e^{-2t}\big(\cos(2t)-\sin(2t)\big)$ that

$$\begin{aligned}y(t) = \mathcal{L}^{-1}[Y(s)](t) &= \mathcal{L}^{-1}\left[\frac{e^{-2s}s}{s^2+4s+8}\right](t) = u(t-2)\mathcal{L}^{-1}\left[\frac{s}{s^2+4s+8}\right](t-2)\\ &= u(t-2)e^{-2(t-2)}\big(\cos(2(t-2))-\sin(2(t-2))\big)\,.\end{aligned}$$

(10) Consider two interconnected tanks filled with brine (salt water). The first tank contains 80 liters and the second contains 30 liters. Brine flows with a concentration of 3 grams of salt per liter flows into the first tank at a rate of 2 liters per hour. Well stirred brine flows from the first tank to the second at a rate of 6 liters per hour, from the second to the first at a rate of 4 liters per hour, and from the second into a drain at a rate of 2 liters per hour. At $t=0$ there are 7 grams of salt in the first tank and 25 grams in the second. Give an initial-value problem that governs the amount of salt in each tank as a function of time.

Solution: The rates work out so there will always be 80 liters of brine in the first tank and 30 liters in the second. Let $S_1(t)$ be the grams of salt in the first tank and $S_2(t)$ be the grams of salt in the second tank. These are governed by the initial-value problem

$$\begin{aligned}\frac{\mathrm{d}S_1}{\mathrm{d}t} &= 3\cdot 2 + \frac{S_2}{30}4 - \frac{S_1}{80}6\,, \qquad & S_1(0) &= 7\,,\\ \frac{\mathrm{d}S_2}{\mathrm{d}t} &= \frac{S_1}{80}6 - \frac{S_2}{30}4 - \frac{S_2}{30}2\,, \qquad & S_2(0) &= 25\,.\end{aligned}$$

You could leave the answer in the above form. It can however be simplified to

$$\begin{aligned}\frac{\mathrm{d}S_1}{\mathrm{d}t} &= 6 + \tfrac{2}{15}S_2 - \tfrac{3}{40}S_1\,, \qquad & S_1(0) &= 7\,,\\ \frac{\mathrm{d}S_2}{\mathrm{d}t} &= \tfrac{3}{40}S_1 - \tfrac{1}{5}S_2\,, \qquad & S_2(0) &= 25\,.\end{aligned}$$

(11) Consider the real vector-valued functions $\mathbf{x}_1(t) = \begin{pmatrix}1\\ t\end{pmatrix}$, $\mathbf{x}_2(t) = \begin{pmatrix}t^3\\ 3+t^4\end{pmatrix}$.

(a) Compute the Wronskian $W[\mathbf{x}_1,\mathbf{x}_2](t)$.

(b) Suppose that $\mathbf{x}_1$ and $\mathbf{x}_2$ comprise a fundamental set of solutions to the linear system $\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$. Give a general solution to this system.

Solution (a). The Wronskian is given by

$$W[\mathbf{x}_1, \mathbf{x}_2](t) = \det\begin{pmatrix} 1 & t^3 \\ t & 3+t^4 \end{pmatrix} = 1 \cdot (3+t^4) - t \cdot t^3 = 3 + t^4 - t^4 = 3\,.$$

Solution (b). Because $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$ is a fundamental set of solutions for the linear system whose coefficient matrix is $\mathbf{A}(t)$, a general solution is given by

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) = c_1\begin{pmatrix} 1 \\ t \end{pmatrix} + c_2\begin{pmatrix} t^3 \\ 3+t^4 \end{pmatrix}\,.$$

(12) Give a general real vector-valued solution of the linear planar system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ for

$$\text{(a)} \quad \mathbf{A} = \begin{pmatrix} 6 & 4 \\ 4 & 0 \end{pmatrix}\,, \qquad\qquad \text{(b)} \quad \mathbf{A} = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}\,.$$

Solution (a). The characteristic polynomial of $\mathbf{A}$ is

$$\begin{aligned} p(z) &= z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) \\ &= z^2 - 6z - 16 = (z-3)^2 - 25 = (z-3)^2 - 5^2\,. \end{aligned}$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which are 3 ± 5, or simply -2 and 8. Therefore we have

$$\begin{aligned} e^{t\mathbf{A}} &= e^{3t}\left[\cosh(5t)\mathbf{I} + \frac{\sinh(5t)}{5}(\mathbf{A} - 3\mathbf{I})\right] \\ &= e^{3t}\left[\cosh(5t)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\sinh(5t)}{5}\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}\right] \\ &= e^{3t}\begin{pmatrix} \cosh(5t) + \frac{3}{5}\sinh(5t) & \frac{4}{5}\sinh(5t) \\ \frac{4}{5}\sinh(5t) & \cosh(5t) - \frac{3}{5}\sinh(5t) \end{pmatrix}\,. \end{aligned}$$

Therefore a general solution is given by

$$\mathbf{x}(t) = e^{t\mathbf{A}}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = c_1 e^{3t}\begin{pmatrix} \cosh(5t) + \frac{3}{5}\sinh(5t) \\ \frac{4}{5}\sinh(5t) \end{pmatrix} + c_2 e^{3t}\begin{pmatrix} \frac{4}{5}\sinh(5t) \\ \cosh(5t) - \frac{3}{5}\sinh(5t) \end{pmatrix}\,.$$

Alternative Solution (a). The characteristic polynomial of $\mathbf{A}$ is

$$\begin{aligned} p(z) &= z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) \\ &= z^2 - 6z - 16 = (z-3)^2 - 25 = (z-3)^2 - 5^2\,. \end{aligned}$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which are 3 ± 5, or simply -2 and 8. Because

$$\mathbf{A} + 2\mathbf{I} = \begin{pmatrix} 8 & 4 \\ 4 & 2 \end{pmatrix}\,, \qquad \mathbf{A} - 8\mathbf{I} = \begin{pmatrix} -2 & 4 \\ 4 & -8 \end{pmatrix}\,,$$

we see that $\mathbf{A}$ has the eigenpairs

$$\left(-2\,, \begin{pmatrix} 1 \\ -2 \end{pmatrix}\right)\,, \qquad \left(8\,, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right)\,.$$

Form these eigenpairs we construct the solutions

$$\mathbf{x}_1(t) = e^{-2t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}, \qquad \mathbf{x}_2(t) = e^{8t}\begin{pmatrix} 2 \\ 1 \end{pmatrix},$$

Therefore a general solution is

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) = c_1 e^{-2t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 e^{8t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Solution (b). The characteristic polynomial of $\mathbf{A}$ is

$$\begin{aligned} p(z) &= z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) \\ &= z^2 - 2z + 5 = (z-1)^2 + 4 = (z-1)^2 + 2^2 . \end{aligned}$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which are $1 \pm i2$. Therefore we have

$$\begin{aligned} e^{t\mathbf{A}} &= e^t\left[\cos(2t)\mathbf{I} + \frac{\sin(2t)}{2}(\mathbf{A} - \mathbf{I})\right] \\ &= e^t\left[\cos(2t)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\sin(2t)}{2}\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}\right] \\ &= e^t\begin{pmatrix} \cos(2t) & \sin(2t) \\ -\sin(2t) & \cos(2t) \end{pmatrix}. \end{aligned}$$

Therefore a general solution is given by

$$\mathbf{x}(t) = e^{t\mathbf{A}}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = c_1 e^t\begin{pmatrix} \cos(2t) \\ -\sin(2t) \end{pmatrix} + c_2 e^t\begin{pmatrix} \sin(2t) \\ \cos(2t) \end{pmatrix}.$$

Alternative Solution (b). The characteristic polynomial of $\mathbf{A}$ is

$$\begin{aligned} p(z) &= z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) \\ &= z^2 - 2z + 5 = (z-1)^2 + 4 = (z-1)^2 + 2^2 . \end{aligned}$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which are $1 \pm i2$. Because

$$\mathbf{A} - (1+i2)\mathbf{I} = \begin{pmatrix} -i2 & 2 \\ -2 & -i2 \end{pmatrix}, \qquad \mathbf{A} - (1-i2)\mathbf{I} = \begin{pmatrix} i2 & 2 \\ -2 & i2 \end{pmatrix},$$

we see that $\mathbf{A}$ has the eigenpairs

$$\left(1+i2\,, \begin{pmatrix} 1 \\ i \end{pmatrix}\right), \qquad \left(1-i2\,, \begin{pmatrix} -i \\ 1 \end{pmatrix}\right).$$

Because

$$e^{(1+i2)t}\begin{pmatrix} 1 \\ i \end{pmatrix} = e^t\begin{pmatrix} \cos(2t) + i\sin(2t) \\ -\sin(2t) + i\cos(2t) \end{pmatrix},$$

two real solutions of the system are

$$\mathbf{x}_1(t) = e^t\begin{pmatrix} \cos(2t) \\ -\sin(2t) \end{pmatrix}, \qquad \mathbf{x}_2(t) = e^t\begin{pmatrix} \sin(2t) \\ \cos(2t) \end{pmatrix}.$$

Therefore a general solution is

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) = c_1 e^t\begin{pmatrix} \cos(2t) \\ -\sin(2t) \end{pmatrix} + c_2 e^t\begin{pmatrix} \sin(2t) \\ \cos(2t) \end{pmatrix}.$$

(13) What answer will be produced by the following MATLAB command?

$$>> \mathrm{A} = [1\ 4;\ 3\ 2];\ [\mathrm{vect},\ \mathrm{val}] = \mathrm{eig}(\mathrm{sym}(\mathrm{A}))$$

You do not have to give the answer in MATLAB format.

Solution. The MATLAB command will produce the eigenpairs of $\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}$. The characteristic polynomial of $\mathbf{A}$ is

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 3z - 10 = (z-5)(z+2)\,,$$

so its eigenvalues are 5 and -2. Because

$$\mathbf{A} - 5\mathbf{I} = \begin{pmatrix} -4 & 4 \\ 3 & -3 \end{pmatrix}, \qquad \mathbf{A} + 2\mathbf{I} = \begin{pmatrix} 3 & 4 \\ 3 & 4 \end{pmatrix},$$

we can read off that the eigenpairs are

$$\left(5\,,\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right), \qquad \left(-2\,,\begin{pmatrix} -4 \\ 3 \end{pmatrix}\right).$$

(14) A real 2×2 matrix $\mathbf{A}$ has eigenvalues 2 and -1 with associated eigenvectors

$$\begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

(a) Give a general solution to the linear planar system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.
(b) Compute $e^{t\mathbf{A}}$.
(c) Sketch a phase-plane portrait for this system and identify its type. Classify the stability of the origin. Carefully mark all sketched trajectories with arrows!

Solution (a). Use the given eigenpairs to construct the solutions

$$\mathbf{x}_1(t) = e^{2t}\begin{pmatrix} 3 \\ 1 \end{pmatrix}, \qquad \mathbf{x}_2(t) = e^{-t}\begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

Therefore a general solution is

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) = c_1 e^{2t}\begin{pmatrix} 3 \\ 1 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

Solution (b). The matrix $\mathbf{A}$ can be diagonalized as $\mathbf{A} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1}$ where

$$\mathbf{V} = \begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbf{V}^{-1} = \frac{1}{7}\begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}.$$

Then

$$\begin{aligned} e^{t\mathbf{A}} &= \mathbf{V}e^{t\mathbf{D}}\mathbf{V}^{-1} = \frac{1}{7}\begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} e^{2t} & 0 \\ 0 & e^{-t} \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix} \\ &= \frac{1}{7}\begin{pmatrix} 3e^{2t} & -e^{-t} \\ e^{2t} & 2e^{-t} \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 6e^{2t} + e^{-t} & 3e^{2t} - 3e^{-t} \\ 2e^{2t} - 2e^{-t} & e^{2t} + 6e^{-t} \end{pmatrix}. \end{aligned}$$

Alternative Solution (b). By part (a) a fundamental matrix is

$$\mathbf{\Psi}(t) = \begin{pmatrix} \mathbf{x}_1(t) & \mathbf{x}_2(t) \end{pmatrix} = \begin{pmatrix} 3e^{2t} & -e^{-t} \\ e^{2t} & 2e^{-t} \end{pmatrix} .$$

Then

$$\begin{aligned} e^{t\mathbf{A}} &= \mathbf{\Psi}(t)\mathbf{\Psi}(0)^{-1} = \begin{pmatrix} 3e^{2t} & -e^{-t} \\ e^{2t} & 2e^{-t} \end{pmatrix} \begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}^{-1} \\ &= \frac{1}{7} \begin{pmatrix} 3e^{2t} & -e^{-t} \\ e^{2t} & 2e^{-t} \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix} = \frac{1}{7} \begin{pmatrix} 6e^{2t} + e^{-t} & 3e^{2t} - 3e^{-t} \\ 2e^{2t} - 2e^{-t} & e^{2t} + 6e^{-t} \end{pmatrix} . \end{aligned}$$

Solution (c). The matrix $\mathbf{A}$ has two real eigenvalues of opposite sign. The origin is therefore a *saddle* and is thereby *unstable*. There is one trajectory moves away from $(0,0)$ along each half of the line $x = 3y$, and one trajectory moves towards$(0,0)$ along each half of the line $y = -2x$. (These are the lines of eigenvectors.) Every other trajectory sweeps away from the line $y = -2x$ and towards the line $x = 3y$. A phase-plane portrait might be sketched during the review session.

(15) Solve the initial-value problem $\mathbf{x}' = \mathbf{A}\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_I$ and describe how its solution behaves as $t \to \infty$ for the following $\mathbf{A}$ and $\mathbf{x}_I$.

(a) $\mathbf{A} = \begin{pmatrix} 3 & 10 \\ -5 & -7 \end{pmatrix}, \qquad \mathbf{x}_I = \begin{pmatrix} -3 \\ 2 \end{pmatrix}.$

(b) $\mathbf{A} = \begin{pmatrix} 8 & -5 \\ 5 & -2 \end{pmatrix}, \qquad \mathbf{x}_I = \begin{pmatrix} 3 \\ -1 \end{pmatrix}.$

Solution (a). The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} 3 & 10 \\ -5 & -7 \end{pmatrix}$ is

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 + 4z + 29 = (z+2)^2 + 5^2 .$$

Therefore the eigenvlues of $\mathbf{A}$ are $-2 \pm i5$. Then

$$\begin{aligned} e^{t\mathbf{A}} &= e^{-2t} \left[\cos(5t)\mathbf{I} + \frac{\sin(5t)}{5}(\mathbf{A} + 2\mathbf{I}) \right] \\ &= e^{-2t} \left[\cos(5t) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\sin(5t)}{5} \begin{pmatrix} 5 & 10 \\ -5 & -5 \end{pmatrix} \right] \\ &= e^{-2t} \begin{pmatrix} \cos(5t) + \sin(5t) & 2\sin(5t) \\ -\sin(5t) & \cos(5t) - \sin(5t) \end{pmatrix} . \end{aligned}$$

The solution of the initial-value problem is therefore

$$\begin{aligned} \mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{x}_I &= e^{-2t} \begin{pmatrix} \cos(5t) + \sin(5t) & 2\sin(5t) \\ -\sin(5t) & \cos(5t) - \sin(5t) \end{pmatrix} \begin{pmatrix} -3 \\ 2 \end{pmatrix} \\ &= e^{-2t} \begin{pmatrix} -3\cos(5t) + \sin(5t) \\ 2\cos(5t) + \sin(5t) \end{pmatrix} . \end{aligned}$$

This solution decays to zero as $t \to \infty$.

Alternative Solution (a). The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} 3 & 10 \\ -5 & -7 \end{pmatrix}$ is

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 + 4z + 29 = (z+2)^2 + 5^2 \,.$$

Therefore the eigenvlues of $\mathbf{A}$ are $-2 \pm i5$. Because

$$\mathbf{A} - (-2+i5)\mathbf{I} = \begin{pmatrix} 5-i5 & 10 \\ -5 & -5-i5 \end{pmatrix},$$

we can read off that $\mathbf{A}$ has the conjugate eigenpairs

$$\left(-2+i5\,, \begin{pmatrix} 1+i \\ -1 \end{pmatrix}\right), \qquad \left(-2-i5\,, \begin{pmatrix} 1-i \\ -1 \end{pmatrix}\right).$$

Because

$$\begin{aligned} e^{-2t+i5t}\begin{pmatrix} 1+i \\ -1 \end{pmatrix} &= e^{-2t}\big(\cos(5t) + i\sin(5t)\big)\begin{pmatrix} 1+i \\ -1 \end{pmatrix} \\ &= e^{-2t}\begin{pmatrix} \cos(5t) - \sin(5t) + i\cos(5t) + i\sin(5t) \\ -\cos(5t) - i\sin(5t) \end{pmatrix}, \end{aligned}$$

a fundamental set of real solutions is

$$\mathbf{x}_1(t) = e^{-2t}\begin{pmatrix} \cos(5t) - \sin(5t) \\ -\cos(5t) \end{pmatrix}, \qquad \mathbf{x}_2(t) = e^{-2t}\begin{pmatrix} \cos(5t) + \sin(5t) \\ -\sin(5t) \end{pmatrix}.$$

Then a fundamental matrix $\mathbf{\Psi}(t)$ is given by

$$\mathbf{\Psi}(t) = \big(\mathbf{x}_1(t) \quad \mathbf{x}_2(t)\big) = e^{-2t}\begin{pmatrix} \cos(5t) - \sin(5t) & \cos(5t) + \sin(5t) \\ -\cos(5t) & -\sin(5t) \end{pmatrix}.$$

Because

$$\mathbf{\Psi}(0)^{-1} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}^{-1} = \frac{1}{1}\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix},$$

we see that

$$\begin{aligned} e^{t\mathbf{A}} = \mathbf{\Psi}(t)\mathbf{\Psi}(0)^{-1} &= e^{-2t}\begin{pmatrix} \cos(5t) - \sin(5t) & \cos(5t) + \sin(5t) \\ -\cos(5t) & -\sin(5t) \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \\ &= e^{-2t}\begin{pmatrix} \cos(5t) + \sin(5t) & 2\sin(5t) \\ -\sin(5t) & \cos(5t) - \sin(5t) \end{pmatrix}. \end{aligned}$$

Therefore the solution of the initial-value problem is

$$\begin{aligned} \mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{x}_I &= e^{-2t}\begin{pmatrix} \cos(5t) + \sin(5t) & 2\sin(5t) \\ -\sin(5t) & \cos(5t) - \sin(5t) \end{pmatrix}\begin{pmatrix} -3 \\ 2 \end{pmatrix} \\ &= e^{-2t}\begin{pmatrix} -3\cos(5t) + \sin(5t) \\ 2\cos(5t) + \sin(5t) \end{pmatrix}. \end{aligned}$$

This solution decays to zero as $t \to \infty$.

Remark. After you have constructed the fundmental set of solutions $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$, you could also have solved the initial-value problem by finding constants c_1 and c_2 such that $\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t)$ satisfies the initial condition. Had you done this using the $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ constructed above, you would have found that $c_1 = -2$ and $c_2 = -1$.

Solution (b). The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} 8 & -5 \\ 5 & -2 \end{pmatrix}$ is

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 6z + 9 = (z-3)^2 .$$

The ony eigenvalue of $\mathbf{A}$ is 3. Then

$$\begin{aligned} e^{t\mathbf{A}} = e^{3t}\big[\mathbf{I} + t\,(\mathbf{A} - 3\mathbf{I})\big] &= e^{3t}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t\begin{pmatrix} 5 & -5 \\ 5 & -5 \end{pmatrix}\right] \\ &= e^{3t}\begin{pmatrix} 1+5t & -5t \\ 5t & 1-5t \end{pmatrix} . \end{aligned}$$

Therefore the solution of the initial-value problem is

$$\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{x}_I = e^{3t}\begin{pmatrix} 1+5t & -5t \\ 5t & 1-5t \end{pmatrix}\begin{pmatrix} 3 \\ -1 \end{pmatrix} = e^{3t}\begin{pmatrix} 3+20t \\ -1+20t \end{pmatrix} .$$

This solution grows like $20t\,e^{3t}$ as $t \to \infty$.

Alternative Solution (b). The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} 8 & -5 \\ 5 & -2 \end{pmatrix}$ is

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 6z + 9 = (z-3)^2 .$$

The only eigenvalue of $\mathbf{A}$ is 3. Because

$$\mathbf{A} - 3\mathbf{I} = \begin{pmatrix} 5 & -5 \\ 5 & -5 \end{pmatrix} ,$$

we can read off that $\mathbf{A}$ has the eigenpair

$$\left(3\,, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) .$$

We can use this eigenpair to construct the solution

$$\mathbf{x}_1(t) = e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} .$$

A second solution can be constructed by

$$\mathbf{x}_2(t) = e^{3t}\mathbf{w} + t\,e^{3t}(\mathbf{A} - 3\mathbf{I})\mathbf{w} ,$$

where $\mathbf{w}$ is any nonzero vector that is not an eigenvector associated with 3. For example, taking $\mathbf{w} = \begin{pmatrix} 1 & 0 \end{pmatrix}^T$ yields

$$\mathbf{x}_2(t) = e^{3t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + t\,e^{3t}\begin{pmatrix} 5 & -5 \\ 5 & -5 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = e^{3t}\begin{pmatrix} 1+5t \\ 5t \end{pmatrix} .$$

Then a fundamental matrix $\mathbf{\Psi}(t)$ is given by

$$\mathbf{\Psi}(t) = \begin{pmatrix} \mathbf{x}_1(t) & \mathbf{x}_2(t) \end{pmatrix} = e^{3t}\begin{pmatrix} 1 & 1+5t \\ 1 & 5t \end{pmatrix} .$$

Because

$$\mathbf{\Psi}(0)^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{-1} = \frac{1}{-1}\begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} ,$$

we see that

$$e^{t\mathbf{A}} = \mathbf{\Psi}(t)\mathbf{\Psi}(0)^{-1} = \begin{pmatrix} 1 & 1+5t \\ 1 & 5t \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1+5t & -5t \\ 5t & 1-5t \end{pmatrix}.$$

Therefore the solution of the initial-value problem is

$$\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{x}_I = e^{3t} \begin{pmatrix} 1+5t & -5t \\ 5t & 1-5t \end{pmatrix} \begin{pmatrix} 3 \\ -1 \end{pmatrix} = e^{3t} \begin{pmatrix} 3+20t \\ -1+20t \end{pmatrix}.$$

This solution grows like $20t\, e^{3t}$ as $t \to \infty$.

Remark. After you have constructed the fundmental set of solutions $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$, you could also have solved the initial-value problem by finding constants c_1 and c_2 such that $\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t)$ satisfies the initial condition. Had you done this using the $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ constructed above, you would have found that $c_1 = -1$ and $c_2 = 4$.

(16) Consider the nonlinear planar system

$$x' = 2xy\,,$$
$$y' = 9 - 9x - y^2\,.$$

(a) Find all of its equilibrium (critical, stationary) points.
(b) Find a nonconstant function $H(x, y)$ such that every trajectory of the system satisfies $H(x, y) = c$ for some constant c.
(c) Classify the type and stability of each equilibrium (critical, stationary) point.
(d) Sketch the level sets (contour lines) $H(x, y) = c$ for values of c corresponding to each saddle point. Use arrows to indicate the direction of the trajectory along each curve that you sketch!

Solution (a). Stationary points satisfy

$$0 = 2xy\,,$$
$$0 = 9 - 9x - y^2\,.$$

The top equation shows that $x = 0$ or $y = 0$. If $x = 0$ then the bottom equation becomes $0 = 9 - y^2 = (3 - y)(3 + y)$, which shows that either $y = 3$ or $y = -3$. If $y = 0$ then the bottom equation becomes $0 = 9 - 9x = 9(1 - x)$, which shows that $x = 1$. Therefore the stationary points of the system are

$$(0, 3)\,, \qquad (0, -3)\,, \qquad (1, 0)\,.$$

Solution (b). The associated first-order trajectory equation is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{9 - 9x - y^2}{2xy}\,.$$

This equation is not linear or separable. It has the differential form

$$(y^2 + 9x + -9)\,\mathrm{d}x + 2xy\,\mathrm{d}y = 0\,,$$

which is exact because

$$\partial_y(y^2 + 9x - 9) = 2y \qquad = \qquad \partial_x(2xy) = 2y\,.$$

Therefore there exists $H(x, y)$ such that

$$\partial_x H(x, y) = y^2 + 9x - 9\,, \qquad \partial_y H(x, y) = 2xy\,.$$

By integrating the second equation we see that

$$H(x, y) = xy^2 + h(x)\,.$$

When this is substituted into the first equation we find

$$\partial_x H(x, y) = y^2 + h'(x) = y^2 + 9x - 9\,,$$

which implies that $h'(x) = 9x - 9$. By taking $h(x) = \frac{9}{2}x^2 - 9x$ we obtain

$$H(x, y) = xy^2 + \tfrac{9}{2}x^2 - 9x\,.$$

Alternative Solution (b). Notice that

$$\partial_x f(x, y) + \partial_y g(x, y) = \partial_x(2xy) + \partial_y(9 - 9x - y^2) = 2y - 2y = 0\,.$$

Therefore the system has Hamiltonian form with Hamiltonian $H(x, y)$ that satisfies

$$\partial_y H(x, y) = 2xy\,, \qquad -\partial_x H(x, y) = 9 - 9x - y^2\,.$$

By integrating the first equation we see that

$$H(x, y) = xy^2 + h(x)\,.$$

When this is substituted into the second equation we find

$$-\partial_x H(x, y) = -y^2 - h'(x) = 9 - 9x - y^2\,,$$

which implies that $h'(x) = 9x - 9$. By taking $h(x) = \frac{9}{2}x^2 - 9x$ we obtain

$$H(x, y) = xy^2 + \tfrac{9}{2}x^2 - 9x\,.$$

Solution (c). Because

$$\begin{pmatrix} f(x, y) \\ g(x, y) \end{pmatrix} = \begin{pmatrix} 2xy \\ 9 - 9x - y^2 \end{pmatrix}\,,$$

the Jacobian matrix $\mathbf{J}(x, y)$ of partial derivatives is

$$\mathbf{J}(x, y) = \begin{pmatrix} \partial_x f(x, y) & \partial_y f(x, y) \\ \partial_x g(x, y) & \partial_y g(x, y) \end{pmatrix} = \begin{pmatrix} 2y & 2x \\ -9 & -2y \end{pmatrix}\,.$$

Evaluating this matrix at each stationary point yields

$$\mathbf{J}(0, 3) = \begin{pmatrix} 6 & 0 \\ -9 & -6 \end{pmatrix}\,, \qquad \mathbf{J}(0, -3) = \begin{pmatrix} -6 & 0 \\ -9 & 6 \end{pmatrix}\,, \qquad \mathbf{J}(1, 0) = \begin{pmatrix} 0 & 2 \\ -9 & 0 \end{pmatrix}\,.$$

- Because the matrix $\mathbf{J}(0, 3)$ is lower triangular, we can read off that its eigenvalues are 6 and -6. Because these are real with opposite signs, the stationary point $(0, 3)$ is a *saddle* and thereby is *unstable*.
- Because the matrix $\mathbf{J}(0, -3)$ is lower triangular, we can read off that its eigenvalues are -6 and 6. Because these are real with opposite signs, the stationary point $(0, -3)$ is a *saddle* and thereby is *unstable*.

- The characteristic polynomial of the matrix $\mathbf{J}(1,0)$ is
$$p(z) = z^2 + 18\,,$$
so the matrix $\mathbf{J}(1,0)$ has eigenvalues $\pm i\sqrt{18}$. Because these are immaginary and the system has an integral while the lower left entry of $\mathbf{J}(1,0)$ is negative, the stationary point $(1,0)$ is a *clockwise center* and thereby is *stable*.

Alternative Solution (c). If you saw that the system has Hamiltonian form with Hamiltonian $H(x,y)$ from part (b) then you can take this approach. The Hessian matrix $\mathbf{H}(x,y)$ of second partial derivatives of the Hamiltonian $H(x,y)$ is
$$\mathbf{H}(x,y) = \begin{pmatrix} \partial_{xx}H(x,y) & \partial_{xy}H(x,y) \\ \partial_{yx}H(x,y) & \partial_{yy}H(x,y) \end{pmatrix} = \begin{pmatrix} 9 & 2y \\ 2y & 2x \end{pmatrix}.$$
Evaluating this at the stationary points yields
$$\mathbf{H}(0,3) = \begin{pmatrix} 9 & 6 \\ 6 & 0 \end{pmatrix}, \qquad \mathbf{H}(0,-3) = \begin{pmatrix} 9 & -6 \\ -6 & 0 \end{pmatrix}, \qquad \mathbf{H}(1,0) = \begin{pmatrix} 9 & 0 \\ 0 & 2 \end{pmatrix}.$$

- The characteristic polynomial of the matrix $\mathbf{H}(0,3)$ is
$$p(z) = z^2 - 9z - 36 = (z-12)(z+3)\,.$$
Therefore the matrix $\mathbf{H}(0,3)$ has eigenvalues 12 and -3. Because these have different signs, the stationary point $(0,3)$ is a *saddle* and thereby is *unstable*.
- The characteristic polynomial of the matrix $\mathbf{H}(0,-3)$ is
$$p(z) = z^2 - 9z - 36 = (z-12)(z+3)\,.$$
Therefore the matrix $\mathbf{H}(0,-3)$ has eigenvalues 12 and -3. Because these have different signs, the stationary point $(0,-3)$ is a *saddle* and thereby is *unstable*.
- Because the matrix $\mathbf{H}(1,0)$ is diagonal, we can read off that its eigenvalues are 9 and 2. Because these are both positive, the stationary point $(1,0)$ is a *clockwise center* and thereby is *stable*.

Solution (d). The saddle points are $(0,3)$ and $(0,-3)$. Because
$$H(0,3) = H(0,-3) = 0\cdot(\pm 3)^2 + \tfrac{9}{2}\cdot 0^2 - 9\cdot 0 = 0\,.$$
Hence, the level set corresponding to these saddle points is
$$0 = xy^2 + \tfrac{9}{2}x^2 - 9x = (y^2 + \tfrac{9}{2}x - 9)x\,.$$
The points on this set must satisfy either $y^2 + \frac{9}{2}x - 9 = 0$ or $x = 0$. Therefore the level set is the union of the parabola $x = 2 - \frac{2}{9}y^2$ and the y-axis.

Along the y-axis ($x=0$) the y' equation reduces to $y' = 9 - y^2 = (3-y)(3+y)$, whereby the arrows point towards $(0,3)$ and away from $(0,-3)$. Along the parabola $x = 2 - \frac{2}{9}y^2$ the arrows point away from $(0,3)$ and towards $(0,-3)$ because they are saddle points.

(17) Consider the nonlinear planar system
$$x' = -5y\,,$$
$$y' = x - 4y - x^2\,.$$

(a) Find all of its equilibrium (critical, stationary) points.

(b) Compute the Jacobian matrix (derivative matrix, linearization coefficient matrix) at each equilibrium (critical, stationary) point.
(c) Classify the type and stability of each equilibrium (critical, stationary) point.
(d) Sketch a plausible global phase-plane portrait. Use arrows to indicate the direction of the trajectory along each curve that you sketch!

Solution (a). Stationary points satisfy

$$0 = -5y\,, \qquad 0 = x - 4y - x^2\,.$$

The first equation implies $y = 0$, whereby the second equation becomes $0 = x - x^2 = x(1-x)$, which implies either $x = 0$ or $x = 1$. All the stationary points of the system are therefore

$$(0,0)\,, \qquad (1,0)\,.$$

Solution (b). Because

$$\begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix} = \begin{pmatrix} -5y \\ x - 4y - x^2 \end{pmatrix}\,,$$

the Jacobian matrix of partial derivatives is

$$\begin{pmatrix} \partial_x f(x,y) & \partial_y f(x,y) \\ \partial_x g(x,y) & \partial_y g(x,y) \end{pmatrix} = \begin{pmatrix} 0 & -5 \\ 1 - 2x & -4 \end{pmatrix}\,.$$

Evaluating this matrix at each stationary point yields the coefficient matrices

$$\mathbf{A} = \begin{pmatrix} 0 & -5 \\ 1 & -4 \end{pmatrix} \quad \text{at } (0,0)\,, \qquad \mathbf{A} = \begin{pmatrix} 0 & -5 \\ -1 & -4 \end{pmatrix} \quad \text{at } (1,0)\,.$$

Solution (c). The coefficient matrix $\mathbf{A}$ at $(0,0)$ has eigenvalues that satisfy

$$0 = \det(z\mathbf{I} - \mathbf{A}) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 + 4z + 5 = (z+2)^2 + 1^2\,.$$

The eigenvalues are thereby $-2 \pm i$. Because $a_{21} = 1 > 0$, the stationary point $(0,0)$ is therefore a *counterclockwise spiral sink*, which is *asymptotically stable* or *attracting*. This is one of the generic types, so it describes the phase-plane portrait of the nonlinear system near $(0,0)$.

The coefficient matrix $\mathbf{A}$ at $(1,0)$ has eigenvalues that satisfy

$$0 = \det(z\mathbf{I} - \mathbf{A}) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 + 4z - 5 = (z+2)^2 - 3^2\,.$$

The eigenvalues are thereby -2 ± 3, or simply -5 and 1. The stationary point $(1,0)$ is therefore a *saddle*, which is *unstable*. This is one of the generic types, so it describes the phase-plane portrait of the nonlinear system near $(1,0)$.

Solution (d). The nullcline for x' is the line $y = 0$. This line partitions the plane into regions where x is increasing or decreasing as t increases. The nullcline for y' is the parabola $y = \frac{1}{4}(x - x^2)$. This curve partitions the plane into regions where y is increasing or decreasing as t increases. Neither of these nullclines is invariant.

The stationary point $(0,0)$ is a *counterclockwise spiral sink*.

The stationary point $(1,0)$ is a *saddle*. The coefficient matrix $\mathbf{A}$ has eigenvalues -5 and 1. Because

$$\mathbf{A}+5\mathbf{I}=\begin{pmatrix}5 & -5\\ -1 & 1\end{pmatrix}\,,\qquad \mathbf{A}-\mathbf{I}=\begin{pmatrix}-1 & -5\\ -1 & -5\end{pmatrix}\,,$$

it has the eigenpairs

$$\left(-5\,,\begin{pmatrix}1\\ 1\end{pmatrix}\right)\,,\qquad \left(1\,,\begin{pmatrix}-5\\ 1\end{pmatrix}\right)$$

Near $(1,0)$ there is one trajectory that emerges from $(1,0)$ tangent to each side of the line $x=1-5y$. There is also one trajectory that approaches $(1,0)$ tangent to each side of the line $y=x-1$. These trajectories are separatrices. A global phase-plane portrait might be sketched during the review session.

Remark. The global phase-plane portrait becomes clearer if you are able to observe that $H(x,y)=\frac{1}{2}x^2+\frac{5}{2}y^2-\frac{1}{3}x^3$ satisfies

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}H(x,y)&=\partial_x H(x,y)\,x'+\partial_y H(x,y)\,y'\\ &=(x-x^2)(-5y)+5y(x-4y-x^2)=-20y^2\leq 0\,.\end{aligned}$$

The trajectories of the system are thereby seen to cross the contour lines of $H(x,y)$ so as to decrease $H(x,y)$. You would *not* be expected to see this on the Final Exam.

(18) Consider the nonlinear planar system

$$\begin{aligned}x'&=x(3-3x+2y)\,,\\ y'&=y(6-x-y)\,.\end{aligned}$$

Do parts (a-d) as for the previous problem.

(e) Why do solutions that start in the first quadrant stay in the first quadrant?

Solution (a). Stationary points satisfy

$$0=x(3-3x+2y)\,,\qquad 0=y(6-x-y)\,.$$

The first equation implies either $x=0$ or $3-3x+2y=0$, while the second equation implies either $y=0$ or $6-x-y=0$. If $x=0$ and $y=0$ then $(0,0)$ is a stationary point. If $x=0$ and $6-x-y=0$ then $(0,6)$ is a stationary point. If $3-3x+2y=0$ and $y=0$ then $(1,0)$ is a stationary point. If $3-3x+2y=0$ and $6-x-y=0$ then upon solving these equations one finds that $(3,3)$ is a stationary point. All the stationary points of the system are therefore

$$(0,0)\,,\qquad (0,6)\,,\qquad (1,0)\,,\qquad (3,3)\,.$$

Solution (b). Because

$$\begin{pmatrix}f(x,y)\\ g(x,y)\end{pmatrix}=\begin{pmatrix}3x-3x^2+2xy\\ 6y-xy-y^2\end{pmatrix}\,,$$

the matrix of partial derivatives is

$$\begin{pmatrix}\partial_x f(x,y) & \partial_y f(x,y)\\ \partial_x g(x,y) & \partial_y g(x,y)\end{pmatrix}=\begin{pmatrix}3-6x+2y & 2x\\ -y & 6-x-2y\end{pmatrix}\,.$$

Evaluating this matrix at each stationary point yields the coefficient matrices

$$\mathbf{A} = \begin{pmatrix} 3 & 0 \\ 0 & 6 \end{pmatrix} \quad \text{at } (0,0)\,, \qquad \mathbf{A} = \begin{pmatrix} 15 & 0 \\ -6 & -6 \end{pmatrix} \quad \text{at } (0,6)\,,$$
$$\mathbf{A} = \begin{pmatrix} -3 & 2 \\ 0 & 5 \end{pmatrix} \quad \text{at } (1,0)\,, \qquad \mathbf{A} = \begin{pmatrix} -9 & 6 \\ -3 & -3 \end{pmatrix} \quad \text{at } (3,3)\,.$$

Solution (c). The coefficient matrix $\mathbf{A}$ at $(0,0)$ is diagonal, so you can read-off its eigenvalues as 3 and 6. The stationary point $(0,0)$ is thereby a *nodal source*, which is *unstable* (or even better is *repelling*). This is one of the generic types, so it describes the phase-plane portrait of the nonlinear system near $(0,0)$.

The coefficient matrix $\mathbf{A}$ at $(0,6)$ is triangular, so you can read-off its eigenvalues as -6 and 15. The stationary point $(0,6)$ is thereby a *saddle*, which is *unstable*. This is one of the generic types, so it describes the phase-plane portrait of the nonlinear system near $(0,6)$.

The coefficient matrix $\mathbf{A}$ at $(1,0)$ is triangular, so you can read-off its eigenvalues as -3 and 5. The stationary point $(1,0)$ is thereby a *saddle*, which is *unstable*. This is one of the generic types, so it describes the phase-plane portrait of the nonlinear system near $(1,0)$.

The coefficient matrix $\mathbf{A}$ at $(3,3)$ has eigenvalues that satisfy

$$0 = \det(z\mathbf{I} - \mathbf{A}) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 + 12z + 45 = (z+6)^2 + 3^2\,.$$

Its eigenvalues are thereby $-6 \pm i3$. Because $a_{21} = -3 < 0$, the stationary point $(3,3)$ is therefore a *clockwise spiral sink*, which is *asymptotically stable* or *attracting*. This is one of the generic types, so it describes the phase-plane portrait of the nonlinear system near $(3,3)$.

Solution (d). The nullclines for x' are the lines $x = 0$ and $3 - 3x + 2y = 0$. These lines partition the plane into regions where x is increasing or decreasing as t increases. The nullclines for y' are the lines $y = 0$ and $6 - x - y = 0$. These lines partition the plane into regions where y is increasing or decreasing as t increases.

Next, observe that the lines $x = 0$ and $y = 0$ are invariant. A trajectory that starts on one of these lines must stay on that line. Along the line $x = 0$ the system reduces to

$$y' = y(6 - y)\,.$$

Along the line $y = 0$ the system reduces to

$$x' = 3x(1 - x)\,.$$

The arrows along these invariant lines can be determined from a phase-line portrait of these reduced systems.

The stationary point $(0,0)$ is a *nodal source*. The coefficient matrix $\mathbf{A}$ has eigenvalues 3 and 6. Because

$$\mathbf{A} - 3\mathbf{I} = \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}\,, \qquad \mathbf{A} - 6\mathbf{I} = \begin{pmatrix} -3 & 0 \\ 0 & 0 \end{pmatrix}\,,$$

it has the eigenpairs

$$\left(3\,,\begin{pmatrix}1\\0\end{pmatrix}\right)\,,\qquad \left(6\,,\begin{pmatrix}0\\1\end{pmatrix}\right)$$

Near there is one trajectory that emerges from $(0,0)$ along each side of the invariant lines $y=0$ and $x=0$. Every other trajectory emerges from $(0,0)$ tangent to the line $y=0$, which is the line corresponding to the eigenvalue with the smaller absolute value.

The stationary point $(0,6)$ is a *saddle*. The coefficient matrix $\mathbf{A}$ has eigenvalues -6 and 15. Because

$$\mathbf{A}+6\mathbf{I}=\begin{pmatrix}21&0\\-6&0\end{pmatrix}\,,\qquad \mathbf{A}-15\mathbf{I}=\begin{pmatrix}0&0\\-6&-21\end{pmatrix}\,,$$

it has the eigenpairs

$$\left(-6\,,\begin{pmatrix}0\\1\end{pmatrix}\right)\,,\qquad \left(15\,,\begin{pmatrix}7\\-2\end{pmatrix}\right)$$

Near $(0,6)$ there is one trajectory that approaches $(0,6)$ along each side of the invariant line $x=0$. There is also one trajectory that emerges from $(0,6)$ tangent to each side of the line $y=6-\frac{2}{7}x$. These trajectories are separatrices.

The stationary point $(1,0)$ is a *saddle*. The coefficient matrix $\mathbf{A}$ has eigenvalues -3 and 5. Because

$$\mathbf{A}+3\mathbf{I}=\begin{pmatrix}0&2\\0&8\end{pmatrix}\,,\qquad \mathbf{A}-5\mathbf{I}=\begin{pmatrix}-8&2\\0&0\end{pmatrix}\,,$$

it has the eigenpairs

$$\left(-3\,,\begin{pmatrix}1\\0\end{pmatrix}\right)\,,\qquad \left(5\,,\begin{pmatrix}1\\4\end{pmatrix}\right)$$

Near $(1,0)$ there is one trajectory that emerges from $(1,0)$ along each side of the invariant line $y=0$. There is also one trajectory that approaches $(1,0)$ tangent to each side of the line $y=4(x-1)$. These trajectories are also separatrices.

Finally, the stationary point $(3,3)$ is a *clockwise spiral sink*. All trajectories in the positive quadrant will spiral into it. A phase-plane global portrait might be sketched during the review session. Be sure your portrait be correct near each stationary point!

Solution (e). Because the lines $x=0$ and $y=0$ are invariant, the uniqueness theorem implies that no other trajectories can cross them.