

# Modeling Portfolios with Risky Assets

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## 1. Introduction: Risk and Reward

Suppose you are considering how to invest in  $N$  risky assets that are traded on a market that had  $D$  trading days last year. (Typically  $D = 255$ .) Let  $s_i(d)$  be the share price of the  $i^{\text{th}}$  asset at the close of the  $d^{\text{th}}$  trading day of the past year, where  $s_i(0)$  is understood to be the share price at the close of the last trading day before the beginning of the past year. We will assume that every  $s_i(d)$  is positive. You would like to use this price history to gain insight into how to manage your portfolio over the coming year.

We will examine the following questions.

*Can stochastic (random, probabilistic) models be built that quantitatively mimic this price history? How can such models be used to help manage a portfolio?*

**Risky Assets.** *The risk associated with an investment is the uncertainty of its outcome. Every investment has risk associated with it.* Hiding your cash under a mattress puts it at greater risk of loss to theft or fire than depositing it in a bank, and is a sure way to not make money. Depositing your cash into an FDIC insured bank account is the safest investment that you can make — the only risk of loss would be to an extreme national calamity. However, a bank account generally will yield a lower return on your investment than any asset that has more risk associated with it. Such assets include stocks (equities), bonds, commodities (gold, oil, corn, etc.), private equity (venture capital), hedge funds, and real estate. With the exception of real estate, it is not uncommon for prices of these assets to fluctuate one to five percent in a day. Any such asset is called a *risky asset*.

**Remark.** *Market forces generally will insure that assets associated with higher potential reward are also associated with greater risk and vice versa.* Investment offers that seem to violate this principle are always scams.

**Return Rates.** The first thing you must understand that the share price of an asset has very little economic significance. This is because the size of your investment in an asset is the same if you own 100 shares worth 50 dollars each or 25 shares worth 200 dollars each. What is economically significant is how much your investment rises or falls in value. Because your investment in asset  $i$  would have changed by the ratio  $s_i(d)/s_i(d-1)$  over the course of day  $d$ , this ratio is economically significant. Rather than use this ratio as the basic variable, it is customary to use the so-called *return rate*, which we define by

$$r_i(d) = D \frac{s_i(d) - s_i(d-1)}{s_i(d-1)}.$$

The factor  $D$  arises because rates in banking, business, and finance are usually given as annual rates expressed in units of either “per annum” or “% per annum.” Because a day is  $\frac{1}{D}$  years the factor of  $D$  makes  $r_i(d)$  a “per annum” rate. It would have to be multiplied by another factor of 100 to make it a “% per annum” rate. We will always work with “per annum” rates.

**Statistical Approach.** *Given the complexity of the dynamics underlying such market fluctuations, we adopt a statistical approach to quantifying their trends and correlations.* More specifically, we will choose statistics computed from selected return rate histories of the relevant assets. We will then use these statistics to calibrate a model that will predict how a set of ideal portfolios might behave in the future.

*The implicit assumption of this approach is that in the future the market will behave statistically as it did in the past.* This means that the data should be drawn from a long enough return rate history to sample most of the kinds of market events that you expect to see in the future. However, the history should not be too long because very old data will not be relevant to the current market. *To strike a balance, we might use the return rate history from the most recent twelve month period, which we dub “the past year”.* For example, if we are planning our portfolio at the beginning of July 2012 then we will use the return rate histories for July 2011 through June 2012. Then  $D$  would be the number of trading days in this period.

## 2. Mean-Variance Models

Suppose that we have the return rate history  $\{r_i(d)\}_{d=1}^{D_h}$  over a period of  $D_h$  trading days. We will use so-called *mean-variance models*, which are calibrated with *means, variances, and covariances*. We assign day  $d$  a weight  $w(d) > 0$  such that the weights  $\{w(d)\}_{d=1}^{D_h}$  satisfy

$$\sum_{d=1}^{D_h} w(d) = 1 .$$

The return rate means and covariances are then given by

$$m_i = \sum_{d=1}^{D_h} w(d) r_i(d) , \quad \bar{w} = \sum_{d=1}^{D_h} w(d)^2 ,$$
$$v_{ij} = \frac{1}{D} \sum_{d=1}^{D_h} \frac{w(d)}{1 - \bar{w}} (r_i(d) - m_i)(r_j(d) - m_j) .$$

The return rate *standard deviation* for asset  $i$  over the year, denoted  $\sigma_i$ , is given by  $\sigma_i = \sqrt{v_{ii}}$ . This is called the *volatility* of asset  $i$ . *Unfortunately,  $m_i$  is commonly called the expected return rate for asset  $i$  even though it is higher than the return rate that most investors will see.*

In practice the history will extend over a period of one to five years. There are many ways to choose the weights  $\{w(d)\}_{d=1}^{D_h}$ . The most common choice is the so-called *uniform weighting*; this gives each day the same weight by setting  $w(d) = 1/D_h$ . On the other hand, we might want to give more weight to more recent data. For example, we can give each trading day a positive weight that depends only on the quarter in which it lies, giving greater weight to more recent quarters. We could also consider giving different weights to different days of the week, but such a complication should be avoided unless it yields a clear benefit.

*You will have greater confidence in  $m_i$  and  $v_{ij}$  when they are relatively insensitive to different choices of  $D_h$  and the weights  $w(d)$ .*

### 3. Markowitz Portfolios

A 1952 paper by Harry Markowitz had enormous influence on the theory and practice of portfolio management and financial engineering ever since. It presented his doctoral dissertation work at the University of Chicago, for which he was awarded the Nobel Prize in Economics in 1990. It was the first work to quantify how diversifying a portfolio can reduce its risk without changing its expected reward.

The value of any portfolio that holds  $n_i(d)$  shares of asset  $i$  at the end of trading day  $d$  is

$$\Pi(d) = \sum_{i=1}^N n_i(d) s_i(d) .$$

If you hold a long position in asset  $i$  then  $n_i(d) > 0$ . If you hold a short position in asset  $i$  then  $n_i(d) < 0$ . If you hold a neutral position in asset  $i$  then  $n_i(d) = 0$ . We will assume that  $\Pi(d) > 0$  for every  $d$ .

Markowitz carried out his analysis on a class of idealized portfolios that are each characterized by a set of real numbers  $\{f_i\}_{i=1}^N$  such that

$$\sum_{i=1}^N f_i = 1 .$$

The portfolio picks  $n_i(d)$  at the beginning at each trading day  $d$  so that

$$\frac{n_i(d) s_i(d-1)}{\Pi(d-1)} = f_i ,$$

where  $n_i(d)$  need not be an integer. We call these *Markowitz portfolios*. The portfolio holds a long position in asset  $i$  if  $f_i > 0$  and holds a short position if  $f_i < 0$ . If every  $f_i$  is nonnegative then  $f_i$  is the fraction of the portfolio's value held in asset  $i$  at the beginning of each day. A Markowitz portfolio will be *self-financing* if we neglect trading costs because

$$\sum_{i=1}^N n_i(d) s_i(d-1) = \Pi(d-1) .$$

**Portfolio Return Rate.** We see from the self-financing property and the relationship between  $n_i(d)$  and  $f_i$  that the return rate  $r(d)$  of a Markowitz portfolio for trading day  $d$  is

$$\begin{aligned} r(d) &= D \frac{\Pi(d) - \Pi(d-1)}{\Pi(d-1)} \\ &= \sum_{i=1}^N D \frac{n_i(d)s_i(d) - n_i(d)s_i(d-1)}{\Pi(d-1)} \\ &= \sum_{i=1}^N \frac{n_i(d)s_i(d-1)}{\Pi(d-1)} D \frac{s_i(d) - s_i(d-1)}{s_i(d-1)} = \sum_{i=1}^N f_i r_i(d). \end{aligned}$$

The return rate  $r(d)$  for the Markowitz portfolio characterized by  $\{f_i\}_{i=1}^N$  is therefore simply the linear combination with coefficients  $f_i$  of the  $r_i(d)$ .

*This relationship makes the class of Markowitz portfolios easy to analyze. We will therefore use Markowitz portfolios to model real portfolios.*

This relationship can be expressed in the compact form

$$r(d) = \mathbf{f}^\top \mathbf{r}(d),$$

where  $\mathbf{f}$  and  $\mathbf{r}(d)$  are the  $N$ -vectors defined by

$$\mathbf{f} = \begin{pmatrix} f_1 \\ \vdots \\ f_N \end{pmatrix}, \quad \mathbf{r}(d) = \begin{pmatrix} r_1(d) \\ \vdots \\ r_N(d) \end{pmatrix}.$$

**Portfolio Statistics.** For the Markowitz portfolio characterized by  $\mathbf{f}$ , the return rate mean  $\mu$  and variance  $v$  can be expressed simply in terms of the  $N$ -vector of return rate means  $\mathbf{m}$  and the  $N \times N$ -matrix of return rate covariances  $\mathbf{V}$  defined by

$$\mathbf{m} = \begin{pmatrix} m_1 \\ \vdots \\ m_N \end{pmatrix}, \quad \mathbf{V} = \begin{pmatrix} v_{11} & \cdots & v_{1N} \\ \vdots & \ddots & \vdots \\ v_{N1} & \cdots & v_{NN} \end{pmatrix}.$$

We can express the calibration of  $\mathbf{m}$  and  $\mathbf{V}$  given the choice of a return rate history  $\{\mathbf{r}(d)\}_{d=1}^{D_h}$  and weights  $\{w(d)\}_{d=1}^{D_h}$  as

$$\mathbf{m} = \sum_{d=1}^{D_h} w(d) \mathbf{r}(d),$$
$$\mathbf{V} = \frac{1}{D} \sum_{d=1}^{D_h} \frac{w(d)}{1 - \bar{w}} (\mathbf{r}(d) - \mathbf{m}) (\mathbf{r}(d) - \mathbf{m})^{\top}.$$

*Recall that ideally  $\mathbf{m}$  and  $\mathbf{V}$  should be insensitive to these choices.*

The return rate mean  $\mu$  and variance  $v$  of a Markowitz portfolio is easy to analyze because  $r(d) = \mathbf{f}^{\top} \mathbf{r}(d)$ , where  $\mathbf{f}$  is independent of  $d$ . We find

$$\mu = \mathbf{f}^{\top} \mathbf{m}, \quad v = \mathbf{f}^{\top} \mathbf{V} \mathbf{f}.$$

Because  $\mathbf{V}$  is positive definite,  $v > 0$ .

**Remark.** Aspects of Markowitz portfolios are unrealistic. These include:

- the fact portfolios can contain fractional shares of any asset;
- the fact portfolios are rebalanced every trading day;
- the fact transaction costs and taxes are neglected;
- the fact dividends and splits are neglected.

By making these simplifications the subsequent analysis becomes easier. The idea is to find the Markowitz portfolio that is best for a given investor. The expectation is that any real portfolio with a distribution close to that for the optimal Markowitz portfolio will perform nearly as well. Consequently, most investors rebalance at most a few times per year, and not every asset is involved each time. Transaction costs and taxes are thereby limited. Similarly, borrowing costs are kept to a minimum by not borrowing often. The return rates can be adjusted to account for dividends and splits.

## 4. Basic Markowitz Portfolio Theory

The 1952 Markowitz paper initiated what subsequently became known as *modern portfolio theory* (MPT). Because 1952 was long ago, this name has begun to look silly and some have taken to calling it Markowitz Portfolio Theory (still MPT), to distinguish it from more modern theories. (Markowitz simply called it portfolio theory, and often made fun of the name it acquired.)

Portfolio theories strive to maximize reward for a given risk — or what is related, minimize risk for a given reward. They do this by quantifying the notions of reward and risk, and identifying a class of idealized portfolios for which an analysis is tractable. Here we present MPT, the first such theory.

*Markowitz chose to use the return rate mean  $\mu$  as the proxy for reward, and the volatility  $\sigma = \sqrt{v}$  as the proxy for risk. He also chose to analyze the class that we have dubbed Markowitz portfolios.* Then for a portfolio of  $N$  risky assets characterized by  $\mathbf{m}$  and  $\mathbf{V}$ , the problem of minimizing risk for a given reward becomes the problem of minimizing

$$\sigma^2 = \mathbf{f}^T \mathbf{V} \mathbf{f}$$

over  $\mathbf{f} \in \mathbb{R}^N$  subject to the constraints

$$\mathbf{1}^T \mathbf{f} = 1, \quad \mathbf{m}^T \mathbf{f} = \mu,$$

where  $\mu$  is given. Here  $\mathbf{1}$  is the  $N$ -vector that has every entry equal to 1. We will assume that  $\mathbf{m}$  and  $\mathbf{1}$  are not co-linear.

Additional constraints can be imposed. For example, if only long positions are to be considered then we must also impose the entrywise constraints  $\mathbf{f} \geq \mathbf{0}$ , where  $\mathbf{0}$  denotes the  $N$ -vector that has every entry equal to 0.

For each  $\mu$  there is a unique minimizer found by Lagrange multipliers to be

$$\mathbf{f}(\mu) = \frac{c - b\mu}{ac - b^2} \mathbf{V}^{-1} \mathbf{1} + \frac{a\mu - b}{ac - b^2} \mathbf{V}^{-1} \mathbf{m},$$

where

$$a = \mathbf{1}^\top \mathbf{V}^{-1} \mathbf{1}, \quad b = \mathbf{1}^\top \mathbf{V}^{-1} \mathbf{m}, \quad c = \mathbf{m}^\top \mathbf{V}^{-1} \mathbf{m}.$$

The associated minimum value of  $\sigma^2$  is

$$\sigma^2 = \mathbf{f}(\mu)^\top \mathbf{V} \mathbf{f}(\mu) = \frac{1}{a} + \frac{a}{ac - b^2} \left( \mu - \frac{b}{a} \right)^2.$$

This is the equation of a hyperbola in the  $\sigma\mu$ -plane. Because volatility is nonnegative, we only consider the right half-plane  $\sigma \geq 0$ . The volatility  $\sigma$  and mean  $\mu$  of any Markowitz portfolio will be a point  $(\sigma, \mu)$  in this half-plane that lies either on or to the right of this hyperbola. Every point  $(\sigma, \mu)$  on this hyperbola in this half-plane represents a unique Markowitz portfolio. These portfolios are called *frontier portfolios*.

We now replace  $a$ ,  $b$ , and  $c$  with the more meaningful *frontier parameters*,

$$\sigma_{mv} = \frac{1}{\sqrt{a}}, \quad \mu_{mv} = \frac{b}{a}, \quad \nu_{as} = \sqrt{\frac{ac - b^2}{a}}.$$

The volatility  $\sigma$  for the frontier portfolio with mean  $\mu$  is then given by

$$\sigma = \sigma_f(\mu) \equiv \sqrt{\sigma_{mv}^2 + \left(\frac{\mu - \mu_{mv}}{\nu_{as}}\right)^2}.$$

*No portfolio has a volatility  $\sigma$  less than  $\sigma_{mv}$ . In other words,  $\sigma_{mv}$  is the minimum volatility attainable by diversification.* The portfolio corresponding to  $(\sigma_{mv}, \mu_{mv})$  is the *minimum volatility portfolio*. Its distribution is given by

$$\mathbf{f}_{mv} = \mathbf{f}(\mu_{mv}) = \mathbf{f}\left(\frac{b}{a}\right) = \frac{1}{a} \mathbf{V}^{-1} \mathbf{1} = \sigma_{mv}^2 \mathbf{V}^{-1} \mathbf{1}.$$

This distribution depends only upon  $\mathbf{V}$ , and is therefore known with greater confidence than any distribution that also depends upon  $\mathbf{m}$ .

The *efficient frontier* is represented in the right-half  $\sigma\mu$ -plane by the upper branch of the frontier hyperbola. It is given as a function of  $\sigma$  by

$$\mu = \mu_{\text{mv}} + \nu_{\text{as}}\sqrt{\sigma^2 - \sigma_{\text{mv}}^2}, \quad \text{for } \sigma > \sigma_{\text{mv}}.$$

This curve is increasing, concave, and emerges vertically upward from the point  $(\sigma_{\text{mv}}, \mu_{\text{mv}})$ . As  $\sigma \rightarrow \infty$  it is asymptotic to the line

$$\mu = \mu_{\text{mv}} + \nu_{\text{as}}\sigma.$$

The *inefficient frontier* is represented in the right-half  $\sigma\mu$ -plane by the lower branch of the frontier hyperbola. It is given as a function of  $\sigma$  by

$$\mu = \mu_{\text{mv}} - \nu_{\text{as}}\sqrt{\sigma^2 - \sigma_{\text{mv}}^2}, \quad \text{for } \sigma > \sigma_{\text{mv}}.$$

This curve is decreasing, convex, and emerges vertically downward from the point  $(\sigma_{\text{mv}}, \mu_{\text{mv}})$ . As  $\sigma \rightarrow \infty$  it is asymptotic to the line

$$\mu = \mu_{\text{mv}} - \nu_{\text{as}}\sigma.$$

**Remark.** The frontier portfolios are independent of the overall market volatility. *Said another way, the frontier portfolios depend only upon the correlations  $c_{ij}$ , the volatility ratios  $\sigma_i/\sigma_j$ , and the means  $m_i$ . Moreover, the minimum volatility portfolio  $f_{mV}$  depends only upon the correlations and the volatility ratios.* Because markets can exhibit periods of markedly different volatility, it is natural to ask when correlations and volatility ratios might be relatively stable across such periods.

**Remark.** *The efficient frontier quantifies the relationship between risk and reward that we mentioned earlier.* A portfolio management theory typically assumes that investors prefer efficient frontier portfolios and will therefore select an efficient frontier portfolio that is optimal given some measure of the risk aversion of an investor. Our goal is to develop such theories.

## 5. Modeling Portfolios with Risk-Free Assets

Until now we have considered portfolios that contain only risky assets. We now consider two kinds of *risk-free* assets (assets that have no volatility associated with them) that can play a major role in portfolio management.

The first is a *safe investment* that pays dividends at a prescribed interest rate  $\mu_{si}$ . This can be an FDIC insured bank account, or safe securities such as US Treasury Bills, Notes, or Bonds. (US Treasury Bills are most commonly used.) You can only hold a long position in such an asset.

The second is a *credit line* from which you can borrow at a prescribed interest rate  $\mu_{cl}$  up to your credit limit. Such a credit line should require you to put up assets like real estate or part of your portfolio (a *margin*) as collateral from which the borrowed money can be recovered if need be. You can only hold a short position in such an asset.

We will assume that  $\mu_{cl} \geq \mu_{si}$ , because otherwise investors would make money by borrowing at rate  $\mu_{cl}$  in order to invest at the greater rate  $\mu_{si}$ . (Here we are again neglecting transaction costs.) Because free money does not sit around for long, market forces would quickly adjust the rates so that  $\mu_{cl} \geq \mu_{si}$ . In practice,  $\mu_{cl}$  is about three points higher than  $\mu_{si}$ .

We will also assume that a portfolio will not hold a position in both the safe investment and the credit line when  $\mu_{cl} > \mu_{si}$ . To do so would effectively be borrowing at rate  $\mu_{cl}$  in order to invest at the lesser rate  $\mu_{si}$ . While there can be cash-flow management reasons for holding such a position for a short time, it is not a smart long-term position.

*These assumptions imply that every portfolio can be viewed as holding a position in at most one risk-free asset: it can hold either a long position at rate  $\mu_{si}$ , a short position at rate  $\mu_{cl}$ , or a neutral risk-free position.*

**Markowitz Portfolios.** We now extend the notion of Markowitz portfolios to portfolios that might include a single risk-free asset with return rate  $\mu_{rf}$ . Let  $b_{rf}(d)$  denote the balance in the risk-free asset at the start of day  $d$ . For a long position  $\mu_{rf} = \mu_{Si}$  and  $b_{rf}(d) > 0$ , while for a short position  $\mu_{rf} = \mu_{Cl}$  and  $b_{rf}(d) < 0$ .

A *Markowitz portfolio* drawn from one risk-free asset and  $N$  risky assets is uniquely determined by a set of real numbers  $f_{rf}$  and  $\{f_i\}_{i=1}^N$  that satisfies

$$f_{rf} + \sum_{i=1}^N f_i = 1, \quad f_{rf} < 1 \quad \text{if any } f_j \neq 0.$$

The portfolio is rebalanced at the start of each day so that

$$\frac{b_{rf}(d)}{\Pi(d-1)} = f_{rf}, \quad \frac{n_i(d) s_i(d-1)}{\Pi(d-1)} = f_i \quad \text{for } i = 1, \dots, N.$$

The condition  $f_{rf} < 1$  if any  $f_j \neq 0$  states that the safe investment must contain less than the net portfolio value unless it is the entire portfolio.

The portfolio return rate mean  $\mu$  and variance  $v$  are found to be

$$\mu = \mu_{rf} (1 - \mathbf{1}^T \mathbf{f}) + \mathbf{m}^T \mathbf{f}, \quad v = \mathbf{f}^T \mathbf{V} \mathbf{f}.$$

These formulas can be viewed as describing a point that lies on a certain half-line in the  $\sigma\mu$ -plane. Let  $(\sigma, \mu)$  be the point in the  $\sigma\mu$ -plane associated with the Markowitz portfolio characterized by the distribution  $\mathbf{f} \neq \mathbf{0}$ . Notice that  $\mathbf{1}^T \mathbf{f} = 1 - f_{rf} > 0$  because  $\mathbf{f} \neq \mathbf{0}$ . Define

$$\tilde{\mathbf{f}} = \frac{\mathbf{f}}{\mathbf{1}^T \mathbf{f}}.$$

Notice that  $\mathbf{1}^T \tilde{\mathbf{f}} = 1$ . Let  $\tilde{\mu} = \mathbf{m}^T \tilde{\mathbf{f}}$  and  $\tilde{\sigma} = \sqrt{\tilde{\mathbf{f}}^T \mathbf{V} \tilde{\mathbf{f}}}$ . Then  $(\tilde{\sigma}, \tilde{\mu})$  is the point in the  $\sigma\mu$ -plane associated with the Markowitz portfolio without risk-free assets that is characterized by the distribution  $\tilde{\mathbf{f}}$ .

We see that the point  $(\sigma, \mu)$  in the  $\sigma\mu$ -plane lies on the half-line that starts at the point  $(0, \mu_{rf})$  and passes through the point  $(\tilde{\sigma}, \tilde{\mu})$  that corresponds to a portfolio that does not contain the risk-free asset.

Conversely, given any point  $(\tilde{\sigma}, \tilde{\mu})$  corresponding to a Markowitz portfolio that contains no risk-free assets, consider the half-line

$$(\sigma, \mu) = \left( \phi \tilde{\sigma}, (1 - \phi)\mu_{rf} + \phi \tilde{\mu} \right) \quad \text{where } \phi > 0.$$

If a portfolio corresponding to  $(\tilde{\sigma}, \tilde{\mu})$  has distribution  $\tilde{\mathbf{f}}$  then the point on the half-line given by  $\phi$  corresponds to the portfolio with distribution  $\mathbf{f} = \phi \tilde{\mathbf{f}}$ . This portfolio allocates  $1 - \mathbf{1}^T \mathbf{f} = 1 - \phi$  of its value to the risk-free asset. The risk-free asset is held long if  $\phi \in (0, 1)$  and held short if  $\phi > 1$  while  $\phi = 1$  corresponds to a neutral position. We must restrict  $\phi$  to either  $(0, 1]$  or  $[1, \infty)$  depending on whether the risk-free asset is the safe investment or the credit line. This segment of the half-line is called the *capital allocation line* through  $(\tilde{\sigma}, \tilde{\mu})$  associated with the risk-free asset.

*We can therefore use the appropriate capital allocation lines to construct the set of all points in the  $\sigma\mu$ -plane associated with Markowitz portfolios that contain a risk-free asset from the set of all points in the  $\sigma\mu$ -plane associated with Markowitz portfolios that contain no risk-free assets.*

**Efficient Frontier.** We now use the capital allocation line construction to see how the efficient frontier is modified by including risk-free assets. Recall that the efficient frontier for portfolios that contain no risk-free assets is given by

$$\mu = \mu_{mv} + \nu_{as} \sqrt{\sigma^2 - \sigma_{mv}^2} \quad \text{for } \sigma \geq \sigma_{mv}.$$

Every point  $(\tilde{\sigma}, \tilde{\mu})$  on this curve has a unique frontier portfolio associated with it. Because  $\mu_{rf} < \mu_{mv}$  there is a unique half-line that starts at the point  $(0, \mu_{rf})$  and is tangent to this curve. Denote this half-line by

$$\mu = \mu_{rf} + \nu_{tg} \sigma \quad \text{for } \sigma \geq 0.$$

Let  $(\sigma_{tg}, \mu_{tg})$  be the point at which this tangency occurs. The unique frontier portfolio associated with this point is called the *tangency portfolio* associated with the risk-free asset; it has distribution  $\mathbf{f}_{tg} = \mathbf{f}_f(\mu_{tg})$ . Then the appropriate capital allocation line will be part of the efficient frontier.

## 6. Stochastic Models of One Risky Asset

Basic MPT does not give guidance about where to be on the efficient frontier. We now build stochastic models that can be used with basic MPT to address this question. *We will see that maximizing the return rate mean for a given volatility is not the best strategy for maximizing your reward.*

**IID Models for an Asset.** We begin by building models of one risky asset with a share price history  $\{s(d)\}_{d=0}^{D_h}$ . Let  $\{r(d)\}_{d=1}^{D_h}$  be the associated return rate history. Because each  $s(d)$  is positive, each  $r(d)$  lies in the interval  $(-D, \infty)$ . An independent, identically-distributed (IID) model for this history simply independently draws  $D$  random numbers  $\{R(d)\}_{d=1}^{D_h}$  from  $(-D, \infty)$  in accord with a fixed probability density  $q(R)$  over  $(-D, \infty)$ . *Such a model is reasonable if a plot of the points  $\{(d, r(d))\}_{d=1}^{D_h}$  in the  $dr$ -plane appears to be distributed in a way that is uniform in  $d$ .*

**Remark.** IID models are the simplest models that are consistent with the way any portfolio theory is used. Specifically, to use any portfolio theory you must first calibrate a model from historical data. This model is then used to predict how a set of ideal portfolios might behave in the future. Based on these predictions one selects the ideal portfolio that optimizes some objective. *This strategy makes the implicit assumption that in the future the market will behave statistically as it did in the past.*

*This assumption requires the market statistics to be stable relative to its dynamics. But this requires future states to decorrelate from past states.* Markov models are characterized by the assumption that possible future states depend upon the present state but not upon past states, thereby maximizing this decorrelation. IID models are the simplest Markov models.

**Return Rate Probability Densities.** Once you have decided to use an IID model for a particular asset, you might think the next goal is to pick an appropriate probability density  $q(R)$ . However, that is neither practical nor necessary. *Rather, the goal is to identify appropriate statistical information about  $q(R)$  that sheds light on the market. Ideally this information should be insensitive to details of  $q(R)$  within a large class of probability densities.*

Recall that a probability density  $q(R)$  over  $(-D, \infty)$  is a nonnegative integrable function such that

$$\int_{-D}^{\infty} q(R) \, dR = 1 .$$

Because we have been collecting mean and covariance return rate data, we will assume that the probability densities also satisfy

$$\int_{-D}^{\infty} R^2 q(R) \, dR < \infty .$$

The mean  $\mu$  and variance  $\xi$  of  $R$  are then

$$\mu = \text{Ex}(R) = \int_{-D}^{\infty} R q(R) \, dR,$$

$$\xi = \text{Var}(R) = \text{Ex}\left((R - \mu)^2\right) = \int_{-D}^{\infty} (R - \mu)^2 q(R) \, dR.$$

Given  $D$  samples  $\{R(d)\}_{d=1}^{D_h}$  that are drawn from the density  $q(R)$ , we can construct *unbiased estimators* of  $\mu$  and  $\xi$  by

$$\hat{\mu} = \sum_{d=1}^{D_h} w(d) R(d), \quad \hat{\xi} = \sum_{d=1}^{D_h} \frac{w(d)}{1 - \bar{w}} (R(d) - \hat{\mu})^2.$$

Being unbiased estimators means  $\text{Ex}(\hat{\mu}) = \mu$  and  $\text{Ex}(\hat{\xi}) = \xi$ . Moreover,

$$\text{Var}(\hat{\mu}) = \text{Ex}\left((\hat{\mu} - \mu)^2\right) = \bar{w} \xi.$$

This implies that  $\hat{\mu}$  converges to  $\mu$  at the rate  $\sqrt{\bar{w}}$  as  $D_h \rightarrow \infty$ . This rate is fastest for uniform weights, when it is  $D_h^{-\frac{1}{2}}$ .

**Growth Rate Probability Densities.** Given  $D_h$  samples  $\{R(d)\}_{d=1}^{D_h}$  that are drawn from the return rate probability density  $q(R)$ , the associated simulated share prices satisfy

$$S(d) = \left(1 + \frac{1}{D}R(d)\right) S(d-1), \quad \text{for } d = 1, \dots, D_h.$$

If we set  $S(0) = s(0)$  then you can easily see that

$$S(d) = \prod_{d'=1}^d \left(1 + \frac{1}{D}R(d')\right) s(0).$$

The *growth rate*  $X(d)$  is related the return rate  $R(d)$  by

$$e^{\frac{1}{D}X(d)} = 1 + \frac{1}{D}R(d).$$

In other words,  $X(d)$  is the growth rate that yeilds a return rate  $R(d)$  on trading day  $d$ . The formula for  $S(d)$  then takes the form

$$S(d) = \exp\left(\frac{1}{D} \sum_{d'=1}^d X(d')\right) s(0).$$

When  $\{R(d)\}_{d=1}^{D_h}$  is an IID process drawn from the density  $q(R)$  over  $(-D, \infty)$ , it follows that  $\{X(d)\}_{d=1}^{D_h}$  is an IID process drawn from the density  $p(X)$  over  $(-\infty, \infty)$  where  $p(X) dX = q(R) dR$  with  $X$  and  $R$  related by

$$X = D \log\left(1 + \frac{1}{D}R\right), \quad R = D \left(e^{\frac{1}{D}X} - 1\right).$$

More explicitly, the densities  $p(X)$  and  $q(R)$  are related by

$$p(X) = q\left(D \left(e^{\frac{1}{D}X} - 1\right)\right) e^{\frac{1}{D}X}, \quad q(R) = \frac{p\left(D \log\left(1 + \frac{1}{D}R\right)\right)}{1 + \frac{1}{D}R}.$$

Because our models will involve means and variances, we will require that

$$\int_{-\infty}^{\infty} X^2 p(X) dX = \int_{-D}^{\infty} D^2 \log\left(1 + \frac{1}{D}R\right)^2 q(R) dR < \infty,$$

$$\int_{-\infty}^{\infty} D^2 \left(e^{\frac{1}{D}X} - 1\right)^2 p(X) dX = \int_{-D}^{\infty} R^2 q(R) dR < \infty.$$

The big advantage of working with  $p(X)$  rather than  $q(R)$  is the fact that

$$\log\left(\frac{S(d)}{s(0)}\right) = \frac{1}{D} \sum_{d'=1}^d X(d').$$

In other words,  $\log(S(d)/s(0))$  is a sum of an IID process. It is easy to compute the mean and variance of this quantity in terms of those of  $X$ .

The mean  $\gamma$  and variance  $\theta$  of  $X$  are

$$\gamma = \text{Ex}(X) = \int_{-\infty}^{\infty} X p(X) dX,$$

$$\theta = \text{Var}(X) = \text{Ex}\left((X - \gamma)^2\right) = \int_{-\infty}^{\infty} (X - \gamma)^2 p(X) dX.$$

We find that

$$\text{Ex}\left(\log\left(\frac{S(d)}{s(0)}\right)\right) = \frac{d}{D} \gamma, \quad \text{Var}\left(\log\left(\frac{S(d)}{s(0)}\right)\right) = \frac{d}{D^2} \theta.$$

The expected growth and variance of the IID model asset at time  $t = d/D$  years is therefore

$$\text{Ex}\left(\log\left(\frac{S(d)}{s(0)}\right)\right) = \gamma t, \quad \text{Var}\left(\log\left(\frac{S(d)}{s(0)}\right)\right) = \frac{1}{D} \theta t.$$

**Remark.** *The IID model suggests that the growth rate mean  $\gamma$  is a good proxy for the reward of an asset and that  $\sqrt{\frac{1}{D} \theta}$  is a good proxy for its risk. However, these are not the proxies chosen by MPT when it is applied to a portfolio consisting of one risky asset.* Those proxies can be approximated by  $\hat{\mu}$  and  $\sqrt{\frac{1}{D} \hat{\xi}}$  where  $\hat{\mu}$  and  $\hat{\xi}$  are the unbiased estimators of  $\mu$  and  $\xi$  given by

$$\hat{\mu} = \sum_{d=1}^{D_h} w(d) R(d), \quad \hat{\xi} = \sum_{d=1}^{D_h} \frac{w(d)}{1 - \bar{w}} (R(d) - \hat{\mu})^2.$$

## 7. Stochastic Models of Portfolios with Risky Assets

We now consider a market of  $N$  risky assets. Let  $\{s_i(d)\}_{d=0}^{D_h}$  be a share price history of asset  $i$ . Let  $\{r_i(d)\}_{d=1}^{D_h}$  and  $\{x_i(d)\}_{d=1}^{D_h}$  be the associated return rate and growth rate histories, where

$$r_i(d) = D \left( \frac{s_i(d)}{s_i(d-1)} - 1 \right), \quad x_i(d) = D \log \left( \frac{s_i(d)}{s_i(d-1)} \right).$$

Because each  $s_i(d)$  is positive, each  $r_i(d)$  is in  $(-D, \infty)$  while each  $x_i(d)$  is in  $(-\infty, \infty)$ . Let  $\mathbf{r}(d)$  and  $\mathbf{x}(d)$  be the  $N$ -vectors

$$\mathbf{r}(d) = \begin{pmatrix} r_1(d) \\ \vdots \\ r_N(d) \end{pmatrix}, \quad \mathbf{x}(d) = \begin{pmatrix} x_1(d) \\ \vdots \\ x_N(d) \end{pmatrix}.$$

The return rate and growth rate histories can then be expressed simply as  $\{\mathbf{r}(d)\}_{d=1}^{D_h}$  and  $\{\mathbf{x}(d)\}_{d=1}^{D_h}$  respectively.

**IID Models for Markets.** An IID model for this market draws  $D$  random vectors  $\{\mathbf{R}(d)\}_{d=1}^D$  from a fixed probability density  $q(\mathbf{R})$  over  $(-D, \infty)^N$ . Such a model is reasonable if the points  $\{(d, \mathbf{r}(d))\}_{d=1}^D$  are distributed in a way that is uniform in  $d$ . This is hard to visualize when  $N$  is not small. However, a necessary condition for the entire market to have an IID model is that every asset has an IID model. This can be visualized for each asset by plotting the points  $\{(d, r_i(d))\}_{d=1}^D$  in the  $dr$ -plane and seeing if they appear to be distributed in a way that is uniform in  $d$ . Similar visual tests based on pairs of assets can be carried out by plotting the points  $\{(d, r_i(d), r_j(d))\}_{d=1}^D$  in  $\mathbb{R}^3$  with an interactive 3D graphics package.

**Remark.** Such visual tests can only warn you when IID models might not be appropriate for describing the data. There are also statistical tests that can play this role. *There is no visual or statistical test that can insure the validity of using an IID model for a market. However, due to their simplicity, IID models are often used unless there is a good reason not to use them.*

After you have decided to use an IID model for the market, you must gather statistical information about the return rate probability density  $q(\mathbf{R})$ . The mean vector  $\boldsymbol{\mu}$  and covariance matrix  $\boldsymbol{\Xi}$  of  $\mathbf{R}$  are given by

$$\boldsymbol{\mu} = \int \mathbf{R} q(\mathbf{R}) d\mathbf{R}, \quad \boldsymbol{\Xi} = \int (\mathbf{R} - \boldsymbol{\mu})(\mathbf{R} - \boldsymbol{\mu})^T q(\mathbf{R}) d\mathbf{R}.$$

Given any sample  $\{\mathbf{R}(d)\}_{d=1}^{D_h}$  drawn from  $q(\mathbf{R})$ , these have the unbiased estimators

$$\hat{\boldsymbol{\mu}} = \sum_{d=1}^{D_h} w(d) \mathbf{R}(d), \quad \hat{\boldsymbol{\Xi}} = \sum_{d=1}^{D_h} \frac{w(d)}{1 - \bar{w}} (\mathbf{R}(d) - \hat{\boldsymbol{\mu}}) (\mathbf{R}(d) - \hat{\boldsymbol{\mu}})^T.$$

If we assume that such a sample is given by the return rate data  $\{\mathbf{r}(d)\}_{d=1}^{D_h}$  then these estimators are given in terms of the vector  $\mathbf{m}$  and matrix  $\mathbf{V}$  by

$$\hat{\boldsymbol{\mu}} = \mathbf{m}, \quad \hat{\boldsymbol{\Xi}} = D \mathbf{V}.$$

**IID Models for Markowitz Portfolios.** Recall that the value of a portfolio that holds a risk-free balance  $b_{\text{rf}}(d)$  with return rate  $\mu_{\text{rf}}$  and  $n_i(d)$  shares of asset  $i$  during trading day  $d$  is

$$\Pi(d) = b_{\text{rf}}(d) \left(1 + \frac{1}{D} \mu_{\text{rf}}\right) + \sum_{i=1}^N n_i(d) s_i(d).$$

We will assume that  $\Pi(d) > 0$  for every  $d$ . Then the return rate  $r(d)$  and growth rate  $x(d)$  for this portfolio on trading day  $d$  are given by

$$r(d) = D \left( \frac{\Pi(d)}{\Pi(d-1)} - 1 \right), \quad x(d) = D \log \left( \frac{\Pi(d)}{\Pi(d-1)} \right).$$

Recall that the return rate  $r(d)$  for the Markowitz portfolio associated with the distribution  $\mathbf{f}$  can be expressed in terms of the vector  $\mathbf{r}(d)$  as

$$r(d) = (1 - \mathbf{1}^\top \mathbf{f}) \mu_{\text{rf}} + \mathbf{f}^\top \mathbf{r}(d).$$

*This implies that if the underlying market has an IID model with return rate probability density  $q(\mathbf{R})$  then the Markowitz portfolio with distribution  $\mathbf{f}$  has the IID model with return rate probability density  $q_{\mathbf{f}}(R)$  given by*

$$q_{\mathbf{f}}(R) = \int \delta\left(R - (1 - \mathbf{1}^T \mathbf{f})\mu_{r_f} - \mathbf{R}^T \mathbf{f}\right) q(\mathbf{R}) d\mathbf{R}.$$

Here  $\delta(\cdot)$  denotes the *Dirac delta distribution*.

We can compute the mean  $\mu$  and variance  $\xi$  of  $q_{\mathbf{f}}(R)$  to be

$$\mu = (1 - \mathbf{1}^T \mathbf{f})\mu_{r_f} + \mu^T \mathbf{f}, \quad \xi = \mathbf{f}^T \Xi \mathbf{f}.$$

Because  $\mu$  and  $\Xi$  have the unbiased estimators  $\hat{\mu} = \mathbf{m}$  and  $\hat{\Xi} = D\mathbf{V}$ , we see from the foregoing formulas that  $\mu$  and  $\xi$  have the unbiased estimators

$$\hat{\mu} = \mu_{r_f}(1 - \mathbf{1}^T \mathbf{f}) + \mathbf{m}^T \mathbf{f}, \quad \hat{\xi} = D\mathbf{f}^T \mathbf{V} \mathbf{f}.$$

*The idea now is to treat the Markowitz portfolio as a single risky asset that can be modeled by the IID process associated with the growth rate probability density  $p_f(X)$  given by*

$$p_f(X) = q_f\left(D\left(e^{\frac{1}{D}X} - 1\right)\right) e^{\frac{1}{D}X}.$$

The mean  $\gamma$  and variance  $\theta$  of  $X$  are given by

$$\gamma = \int X p_f(X) dX, \quad \theta = \int (X - \gamma)^2 p_f(X) dX.$$

We know from our study of one risky asset that  $\gamma$  is a good proxy for reward, while  $\sqrt{\frac{1}{D}\theta}$  is a good proxy for risk. We therefore would like to estimate  $\gamma$  and  $\theta$  in terms of  $\hat{\mu}$  and  $\hat{\xi}$ .

**Estimators for  $\gamma$  and  $\theta$ .** Introduce the function

$$K(\tau) = \log\left(\text{Ex}\left(e^{\tau X}\right)\right) .$$

Because  $R = D(e^{\frac{1}{D}X} - 1)$  and  $\text{Ex}(e^{\frac{1}{D}X}) = e^{K(\frac{1}{D})}$ , we have

$$\mu = \text{Ex}(R) = D\left(e^{K(\frac{1}{D})} - 1\right) .$$

Because  $R - \mu = D\left(e^{\frac{1}{D}X} - e^{K(\frac{1}{D})}\right)$  and  $\text{Ex}(e^{\frac{2}{D}X}) = e^{K(\frac{2}{D})}$ , we have

$$\xi = \text{Ex}\left((R - \mu)^2\right) = D^2\left(e^{K(\frac{2}{D})} - e^{2K(\frac{1}{D})}\right) .$$

Because  $e^{K(\frac{1}{D})} = 1 + \frac{\mu}{D}$ , we see that

$$e^{K(\frac{2}{D}) - 2K(\frac{1}{D})} = 1 + \frac{\xi}{(D + \mu)^2} .$$

Therefore knowing  $\mu$  and  $\xi$  is equivalent to knowing  $K(\frac{1}{D})$  and  $K(\frac{2}{D})$ .

The function  $K(\tau)$  is the *cumulant generating function* for  $X$  because it recovers the cumulants  $\{\kappa_m\}_{m=1}^{\infty}$  of  $X$  by the formula  $\kappa_m = K^{(m)}(0)$ . In particular, you can check that

$$K'(0) = \gamma, \quad K''(0) = \theta.$$

Because  $K(0) = 0$ , we interpolate the values  $K(0)$ ,  $K(\frac{1}{D})$ , and  $K(\frac{2}{D})$  with a quadratic polynomial to construct an estimator  $\hat{K}(\tau)$  of  $K(\tau)$  as

$$\hat{K}(\tau) = \tau D K\left(\frac{1}{D}\right) + \tau \left(\tau - \frac{1}{D}\right) \frac{D^2}{2} \left(K\left(\frac{2}{D}\right) - 2K\left(\frac{1}{D}\right)\right).$$

We then construct estimators  $\hat{\gamma}$  and  $\hat{\theta}$  by

$$\begin{aligned} \hat{\gamma} &= \hat{K}'(0) = D K\left(\frac{1}{D}\right) - \frac{1}{2}D \left(K\left(\frac{2}{D}\right) - 2K\left(\frac{1}{D}\right)\right) \\ &= D \log\left(1 + \frac{\mu}{D}\right) - \frac{1}{2}D \log\left(1 + \frac{\xi}{(D+\mu)^2}\right), \end{aligned}$$

$$\hat{\theta} = \hat{K}''(0) = D^2 \left(K\left(\frac{2}{D}\right) - 2K\left(\frac{1}{D}\right)\right) = D^2 \log\left(1 + \frac{\xi}{(D+\mu)^2}\right).$$

Upon replacing the  $\mu$  and  $\xi$  in the foregoing estimators for  $\hat{\gamma}$  and  $\hat{\theta}$  with the estimators  $\hat{\mu} = \mu_{rf}(1 - \mathbf{1}^\top \mathbf{f}) + \mathbf{m}^\top \mathbf{f}$  and  $\hat{\xi} = D \mathbf{f}^\top \mathbf{V} \mathbf{f}$ , we obtain the new estimators

$$\hat{\gamma} = D \log\left(1 + \frac{\hat{\mu}}{D}\right) - \frac{1}{2}D \log\left(1 + \frac{D \mathbf{f}^\top \mathbf{V} \mathbf{f}}{(D + \hat{\mu})^2}\right),$$

$$\hat{\theta} = D^2 \log\left(1 + \frac{D \mathbf{f}^\top \mathbf{V} \mathbf{f}}{(D + \hat{\mu})^2}\right).$$

Finally, if we assume  $D$  is large in the sense that

$$\left|\frac{\hat{\mu}}{D}\right| \ll 1, \quad \left|\frac{\mathbf{f}^\top \mathbf{V} \mathbf{f}}{D}\right| \ll 1,$$

then, by keeping the leading order of each term, we arrive at the estimators

$$\hat{\gamma} = \mu_{rf}(1 - \mathbf{1}^\top \mathbf{f}) + \mathbf{m}^\top \mathbf{f} - \frac{1}{2}\mathbf{f}^\top \mathbf{V} \mathbf{f}, \quad \frac{\hat{\theta}}{D} = \mathbf{f}^\top \mathbf{V} \mathbf{f}.$$

**Remark.** The estimators  $\hat{\gamma}$  and  $\hat{\theta}$  given above have at least three potential sources of error:

- the estimators  $\hat{\mu}$  and  $\hat{\xi}$  upon which they are based,
- the interpolant  $\hat{K}(\tau)$  used to estimate  $\gamma$  and  $\theta$  from  $\mu$  and  $\xi$ ,
- the “large  $D$ ” approximation made at the bottom of the previous page.

*These approximations all assume that the return rate distribution for each Markowitz portfolio is described by a density  $q_f(\mathbb{R})$  that is narrow enough for some moment beyond the second to exist. The last approximation also assumes both that  $\frac{1}{D}\mathbf{m}$  and  $\frac{1}{D}\mathbf{V}$  are small and that  $\mathbf{f}$  is not very large. These assumptions should be examined carefully in volatile markets.*

**Remark.** If the Markowitz portfolio specified by  $f$  has growth rates  $X$  that are normally distributed with mean  $\gamma$  and variance  $\theta$  then

$$p_f(X) = \frac{1}{\sqrt{2\pi\theta}} \exp\left(-\frac{(X - \gamma)^2}{2\theta}\right).$$

A direct calculation then shows that

$$\begin{aligned} \text{Ex}(e^{\tau X}) &= \frac{1}{\sqrt{2\pi\theta}} \int \exp\left(-\frac{(X - \gamma)^2}{2\theta} + \tau X\right) dX \\ &= \frac{1}{\sqrt{2\pi\theta}} \int \exp\left(-\frac{(X - \gamma - \theta\tau)^2}{2\theta} + \gamma\tau + \frac{1}{2}\theta\tau^2\right) dX \\ &= \exp\left(\gamma\tau + \frac{1}{2}\theta\tau^2\right), \end{aligned}$$

whereby  $K(\tau) = \log(\text{Ex}(e^{\tau X})) = \gamma\tau + \frac{1}{2}\theta\tau^2$ . In this case we have  $\hat{K}(\tau) = K(\tau)$ , so the estimators  $\hat{\gamma} = \hat{K}'(0)$  and  $\hat{\theta} = \hat{K}''(0)$  are exact. More generally, if  $K(\tau)$  is thrice continuously differentiable over  $[0, \frac{2}{D}]$  then the estimators  $\hat{\gamma}$  and  $\hat{\theta}$  make errors that are  $O(\frac{1}{D^2})$  and  $O(\frac{1}{D})$  respectively.

## 8. Model-Based Objectives

An IID model for the Markowitz portfolio with distribution  $\mathbf{f}$  satisfies

$$\text{Ex}\left(\log\left(\frac{\Pi(d)}{\Pi(0)}\right)\right) = \frac{d}{D}\gamma, \quad \text{Var}\left(\log\left(\frac{\Pi(d)}{\Pi(0)}\right)\right) = \frac{d}{D^2}\theta,$$

where  $\gamma$  and  $\theta$  are estimated from a share price history by

$$\hat{\gamma} = \mu_{\text{rf}}(1 - \mathbf{1}^T \mathbf{f}) + \mathbf{m}^T \mathbf{f} - \frac{1}{2} \mathbf{f}^T \mathbf{V} \mathbf{f}, \quad \frac{\hat{\theta}}{D} = \mathbf{f}^T \mathbf{V} \mathbf{f}.$$

We see that  $\hat{\gamma} t$  is then the estimated expected growth of the IID model while  $\mathbf{f}^T \mathbf{V} \mathbf{f} t$  is its estimated variance at time  $t = d/D$  years.

Our approach to portfolio management will be to select a distribution  $\mathbf{f}$  that maximizes some objective function. Here we develop a family of such objective functions built from  $\hat{\gamma}$  and  $\hat{\theta}$  with the aid of two important tools from probability, the *Law of Large Numbers* and the *Central Limit Theorem*.

**Law of Large Numbers.** Let  $\{X(d)\}_{d=1}^{\infty}$  be any sequence of IID random variables drawn from a probability density  $p(X)$  with mean  $\gamma$  and variance  $\theta > 0$ . Let  $\{Y(d)\}_{d=1}^{\infty}$  be the sequence of random variables defined by

$$Y(d) = \frac{1}{d} \sum_{d'=1}^d X(d') \quad \text{for every } d = 1, \dots, \infty.$$

You can easily check that

$$\text{Ex}(Y(d)) = \gamma, \quad \text{Var}(Y(d)) = \frac{\theta}{d}.$$

Given any  $\delta > 0$  the *Law of Large Numbers* states that

$$\lim_{d \rightarrow \infty} \Pr\{|Y(d) - \gamma| \geq \delta\} = 0.$$

This limit is not uniform in  $\delta$ . Its convergence rate can be estimated by the *Chebyshev Inequality*, which yields the (not uniform in  $\delta$ ) upper bound

$$\Pr\{|Y(d) - \gamma| \geq \delta\} \leq \frac{\text{Var}(Y(d))}{\delta^2} = \frac{1}{\delta^2} \frac{\theta}{d}.$$

**Growth Rate Mean.** Because the value of the associated portfolio is

$$\Pi(d) = \Pi(0) \exp\left(Y(d)\frac{d}{D}\right),$$

we see that  $Y(d)$  is the growth rate of the portfolio at day  $d$ . The Law of Large Numbers implies that  $Y(d)$  is likely to approach  $\gamma$  as  $d \rightarrow \infty$ .

*This suggests that investors whose goal is to maximize the value of their portfolio over an extended period should maximize  $\gamma$ . More precisely, it suggests that such investors should select  $\mathbf{f}$  to maximize the estimator  $\hat{\gamma}$ .*

**Remark.** The suggestion to maximize  $\hat{\gamma}$  rests upon the assumption that the investor will hold the portfolio for an extended period. This is a suitable assumption for most young investors, but not for many old investors. The development of objective functions that are better suited for older investors requires more information about  $Y(d)$  than the Law of Large Numbers provides. However, this additional information can be estimated with the aid of the Central Limit Theorem.

**Central Limit Theorem.** Let  $\{X(d)\}_{d=1}^{\infty}$  be any sequence of IID random variables drawn from a probability density  $p(X)$  with mean  $\gamma$  and variance  $\theta > 0$ . Let  $\{Y(d)\}_{d=1}^{\infty}$  be the sequence of random variables defined by

$$Y(d) = \frac{1}{d} \sum_{d'=1}^d X(d') \quad \text{for every } d = 1, \dots, \infty.$$

Recall that

$$\text{Ex}(Y(d)) = \gamma, \quad \text{Var}(Y(d)) = \frac{\theta}{d}.$$

Now let  $\{Z(d)\}_{d=1}^{\infty}$  be the sequence of random variables defined by

$$Z(d) = \frac{Y(d) - \gamma}{\sqrt{\theta/d}} \quad \text{for every } d = 1, \dots, \infty.$$

These random variables have been normalized so that

$$\text{Ex}(Z(d)) = 0, \quad \text{Var}(Z(d)) = 1.$$

The *Central Limit Theorem* states that as  $d \rightarrow \infty$  the limiting distribution of  $Z(d)$  will be the mean-zero, variance-one normal distribution. Specifically, for every  $\zeta \in \mathbb{R}$  it implies that

$$\lim_{d \rightarrow \infty} \Pr\{Z(d) \geq -\zeta\} = \int_{-\zeta}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}Z^2} dZ.$$

This can be expressed in terms of  $Y(d)$  as

$$\lim_{d \rightarrow \infty} \Pr\left\{Y(d) \geq \gamma - \zeta\sqrt{\theta/d}\right\} = \int_{-\zeta}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}Z^2} dZ.$$

**Remark.** The power of the Central Limit Theorem is that it assumes so little about the underlying probability density  $p(X)$ . Specifically, it assumes that

$$\int_{-\infty}^{\infty} X^2 p(X) dX < \infty,$$

and that

$$0 < \theta = \int_{-\infty}^{\infty} (X - \gamma)^2 p(X) dX, \quad \text{where} \quad \gamma = \int_{-\infty}^{\infty} X p(X) dX.$$

**Remark.** The Central Limit Theorem does not estimate how fast this limit is approached. Any such estimate would require additional assumptions about the underlying probability density  $p(X)$ . It will not be uniform in  $\zeta$ .

**Remark.** In an IID model of a portfolio  $Y(d)$  is the growth rate of the portfolio when it is held for  $d$  days. The Central Limit Theorem shows that as  $d \rightarrow \infty$  the values of  $Y(d)$  become strongly peaked around  $\gamma$ . This behavior seems to be consistent with the idea that a reasonable approach towards portfolio management is to select  $f$  to maximize the estimator  $\hat{\gamma}$ . However, by taking  $\zeta = 0$  we see that the Central Limit Theorem implies

$$\lim_{d \rightarrow \infty} \Pr\{Y(d) \geq \gamma\} = \frac{1}{2}.$$

This shows that in the long run the growth rate of a portfolio will exceed  $\gamma$  with a probability of only  $\frac{1}{2}$ . A conservative investor might want the portfolio to exceed the optimized growth rate with a higher probability.

**Growth Rate Exceeded with Probability.** Let  $\Gamma(\lambda, T)$  be the growth rate exceeded by a portfolio with probability  $\lambda$  at time  $T$  in years. Here we will use the Central Limit Theorem to construct an estimator  $\hat{\Gamma}(\lambda, T)$  of this quantity. We do this by assuming  $T = d/D$  is large enough that we can use the approximation

$$\Pr\left\{Y(d) \geq \gamma - \zeta\sqrt{\theta/d}\right\} \approx \int_{-\zeta}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}Z^2} dZ.$$

Given any probability  $\lambda \in (0, 1)$ , we set

$$\lambda = \int_{-\zeta}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}Z^2} dZ = \int_{-\infty}^{\zeta} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}Z^2} dZ \equiv N(\zeta).$$

Our approximation can then be expressed as

$$\Pr\left\{Y(d) \geq \gamma - \frac{\zeta}{\sqrt{T}}\sigma\right\} \approx \lambda,$$

where  $\sigma = \sqrt{\theta/D}$  and  $\zeta = N^{-1}(\lambda)$ .

Finally, we replace  $\gamma$  and  $\sigma$  in the above approximation by the estimators

$$\hat{\gamma} = \mu_{\text{rf}} (1 - \mathbf{1}^T \mathbf{f}) + \mathbf{m}^T \mathbf{f} - \frac{1}{2} \mathbf{f}^T \mathbf{V} \mathbf{f}, \quad \hat{\sigma} = \sqrt{\mathbf{f}^T \mathbf{V} \mathbf{f}}.$$

This yields the estimator

$$\hat{\Gamma}(\lambda, T) = \hat{\gamma} - \frac{\zeta}{\sqrt{T}} \hat{\sigma} = \hat{\mu} - \frac{1}{2} \hat{\sigma}^2 - \frac{\zeta}{\sqrt{T}} \hat{\sigma},$$

where  $\hat{\mu} = \mu_{\text{rf}} (1 - \mathbf{1}^T \mathbf{f}) + \mathbf{m}^T \mathbf{f}$  and  $\zeta = N^{-1}(\lambda)$ .

**Remark.** The only new assumption we have made in order to construct this estimator is that  $T$  is large enough for the Central Limit Theorem to yield a good approximation of the distribution of growth rates. Investors often choose  $T$  to be the interval at which the portfolio will be rebalanced, regardless of whether  $T$  is large enough for the approximation to be valid. If an investor plans to rebalance once a year then  $T = 1$ , twice a year then  $T = \frac{1}{2}$ , and four times a year then  $T = \frac{1}{4}$ . The smaller  $T$ , the less likely it is that the Central Limit Theorem approximation is valid.

**Risk Aversion.** *The idea now will be to select the admissible Markowitz portfolio that maximizes  $\hat{\Gamma}(\lambda, T)$  given a choice of  $\lambda$  and  $T$  by the investor. In other words, the objective will be to maximize the growth rate that will be exceeded by the portfolio with probability  $\lambda$  when it is held for  $T$  years. Because  $1 - \lambda$  is the fraction of times the investor is willing to experience a downside tail event, the choice of  $\lambda$  measures the *risk aversion* of the investor. More risk averse investors will select a higher  $\lambda$ .*

**Remark.** The risk aversion of an investor generally increases with age. Retirees whose portfolio provides them with an income that covers much of their living expenses will generally be extremely risk averse. Investors within ten years of retirement will be fairly risk averse because they have less time for their nest-egg to recover from any economic downturn. In contrast, young investors can be less risk averse because they have more time to experience economic upturns and because they are typically far from their peak earning capacity.

An investor can simply select  $\zeta$  such that  $\lambda = N(\zeta)$  is a probability that reflects their risk aversion. For example, based on the tabulations

$$N(0) = .5000, \quad N\left(\frac{1}{4}\right) \approx .5987, \quad N\left(\frac{1}{2}\right) \approx .6915, \quad N\left(\frac{3}{4}\right) \approx .7734, \\ N(1) \approx .8413, \quad N\left(\frac{5}{4}\right) \approx .8944, \quad N\left(\frac{3}{2}\right) \approx .9332, \quad N\left(\frac{7}{4}\right) \approx .9505,$$

an investor who is willing to experience a downside tail event roughly

- once every two years might select  $\zeta = 0$ ,
- twice every five years might select  $\zeta = \frac{1}{4}$ ,
- thrice every ten years might select  $\zeta = \frac{1}{2}$ ,
- twice every nine years might select  $\zeta = \frac{3}{4}$ ,
- once every six years might select  $\zeta = 1$ ,
- once every ten years might select  $\zeta = \frac{5}{4}$ ,
- once every fifteen years might select  $\zeta = \frac{3}{2}$ ,
- once every twenty years might select  $\zeta = \frac{7}{4}$ .

**Remark.** *The Central Limit Theorem approximation generally degrades badly as  $\zeta$  increases because  $p(X)$  typically decays much more slowly than a normal density as  $X \rightarrow -\infty$ .* It is therefore a bad idea to pick  $\zeta > 2$  based on this approximation. Fortunately,  $\zeta = \frac{7}{4}$  already corresponds to a fairly conservative investor.

**Remark.** *You should pick a larger value of  $\zeta$  whenever your analysis of the historical data gives you less confidence either in the calibration of  $\mathbb{V}$  and  $\mathbf{m}$  or in the validity of an IID model.*

**Remark.** This approach is similar to something in financial management called *value at risk*. The finance problem is much harder because the time horizon  $T$  considered there is much shorter, typically on the order of days. In that setting the Central Limit Theorem approximation is certainly invalid.

## 9. Model-Based Portfolio Optimization

We now address the problem of how to manage a portfolio that contains  $N$  risky assets along with a risk-free safe investment and possibly a risk-free credit line. Given the mean vector  $\mathbf{m}$ , the covariance matrix  $\mathbf{V}$ , and the risk-free rates  $\mu_{\text{si}}$  and  $\mu_{\text{cl}}$ , the idea is to select the portfolio distribution  $\mathbf{f}$  that maximizes an objective function of the form

$$\hat{\Gamma}(\mathbf{f}) = \hat{\mu} - \frac{1}{2}\hat{\sigma}^2 - \chi \hat{\sigma},$$

where

$$\begin{aligned} \hat{\mu} &= \mu_{\text{rf}} (\mathbf{1} - \mathbf{1}^T \mathbf{f}) + \mathbf{m}^T \mathbf{f}, \\ \hat{\sigma} &= \sqrt{\mathbf{f}^T \mathbf{V} \mathbf{f}}, \end{aligned} \quad \mu_{\text{rf}} = \begin{cases} \mu_{\text{si}} & \text{for } \mathbf{1}^T \mathbf{f} < 1, \\ \mu_{\text{cl}} & \text{for } \mathbf{1}^T \mathbf{f} > 1. \end{cases}$$

Here  $\chi = \zeta / \sqrt{T}$  where  $\zeta \geq 0$  is the *risk aversion coefficient* and  $T > 0$  is a time horizon that is usually the *time to the next portfolio rebalancing*. Both  $\zeta$  and  $T$  are chosen by the investor.

**Reduced Maximization Problem.** *Because frontier portfolios minimize  $\hat{\sigma}$  for a given value of  $\hat{\mu}$ , the optimal  $f$  clearly must be a frontier portfolio. Because the optimal portfolio must also be more efficient than every other portfolio with the same volatility, it must lie on the efficient frontier.*

The efficient frontier is a curve  $\mu = \mu_{\text{ef}}(\sigma)$  in the  $\sigma\mu$ -plane given by an increasing, concave, continuously differentiable function  $\mu_{\text{ef}}(\sigma)$  defined over  $[0, \infty)$ . The problem thereby reduces to finding  $\sigma \in [0, \infty)$  that maximizes

$$\Gamma_{\text{ef}}(\sigma) = \mu_{\text{ef}}(\sigma) - \frac{1}{2}\sigma^2 - \chi\sigma.$$

This function has the continuous derivative  $\Gamma'_{\text{ef}}(\sigma) = \mu'_{\text{ef}}(\sigma) - \sigma - \chi$ . Because  $\mu_{\text{ef}}(\sigma)$  is concave,  $\Gamma'_{\text{ef}}(\sigma)$  is strictly decreasing. In addition  $\Gamma_{\text{ef}}(\sigma) \rightarrow -\infty$  as  $\sigma \rightarrow \infty$ . The maximizer therefore exists and is unique.

This reduced maximization problem can be visualized by considering the family of parabolas parameterized by  $\Gamma$  as

$$\mu = \Gamma + \chi\sigma + \frac{1}{2}\sigma^2.$$

As  $\Gamma$  varies the graph of this parabola shifts up and down in the  $\sigma\mu$ -plane. For some values of  $\Gamma$  the corresponding parabola will intersect the efficient frontier, which is given by  $\mu = \mu_{\text{ef}}(\sigma)$ . There is clearly a maximum such  $\Gamma$ . As the parabola is strictly convex while the efficient frontier is concave, for this maximum  $\Gamma$  the intersection will consist of a single point  $(\sigma_{\text{opt}}, \mu_{\text{opt}})$ . Then  $\sigma = \sigma_{\text{opt}}$  is the maximizer of  $\Gamma_{\text{ef}}(\sigma)$ .

This reduction is appealing because the efficient frontier only depends on general information about an investor, like whether he or she will take short positions. Once it is computed, the problem of maximizing any given  $\hat{\Gamma}(\mathbf{f})$  over all admissible portfolios  $\mathbf{f}$  reduces to the problem of maximizing the associated  $\Gamma_{\text{ef}}(\sigma)$  over all admissible  $\sigma$  — a problem over one variable.

In summary, our approach to portfolio selection has three steps:

1. Choose a return rate history over a given period (say the past year) and calibrate the mean vector  $\mathbf{m}$  and the covariance matrix  $\mathbf{V}$  with it.
2. Given  $\mathbf{m}$ ,  $\mathbf{V}$ ,  $\mu_{SI}$ ,  $\mu_{CI}$ , and any portfolio constraints, compute  $\mu_{ef}(\sigma)$ .
3. Finally, choose  $\chi = \zeta/\sqrt{T}$  and maximize the associated  $\Gamma_{ef}(\sigma)$ ; the maximizer  $\sigma_{opt}$  corresponds to a unique efficient frontier portfolio.

*Rather than fit data to a single model, we considered the whole family of IID models. This gives us greater confidence in the robustness of our results.*

**Remark.** Such models illustrate two basic principles of investing.

*In a bear market the above solution gives an optimal portfolio that is placed largely in the safe investment, but the part of the portfolio placed in risky assets is placed in the most aggressive risky assets.* Such a position allows you to catch market upturns while putting little at risk when the market goes down.

*In a bull market the above solution gives an optimal optimal portfolio that is placed largely in risky assets, but much of it is not placed in the most aggressive risky assets.* Such a position protects you from market downturns while giving up little in returns when the market goes up.

*Many investors will ignore these basic principles and become either overly conservative in a bear market or overly aggressive in a bull market.*

## 10. Conclusion

Such MPT models illustrate three basic principles of portfolio management. (As we have just seen, there are others.)

1. *Diversification reduces the volatility of a portfolio.*
2. *Increased volatility lowers the expected growth rate of a portfolio.*
3. *Diversification raises the expected growth rate of a portfolio.*

**Remark.** The last of these follows from the first two.

One major limitation of the models we have studied is that they assume the validity of an underlying IID model. The truth is that all agents who buy and sell risky assets are influenced by the past. An IID model will be valid when the motives of enough agents are sufficiently diverse and uncorrelated. You can test the validity of this assumption with the historical data. But even when the historical data supports this assumption, you must be on guard for correlations that might arise due to changing circumstances.

Another major limitation is that dependencies between different assets are only captured by the covariances in historical data. Such models can lose validity when a major event occurs that has no analog in the period spanned by the historical data that you used to calibrate your model.

Yet another major limitation is that they assume the probability densities in the underlying IID model are sufficiently narrow that second moments exist. When this assumption is not valid this theory breaks down completely.

*Many common criticisms of MPT are simply wrong.* These include (look up “portfolio theory” on Wikipedia) the following claims:

- it assumes asset returns are normally distributed;
- it assumes markets are efficient;
- it assumes all investors are rational and risk-averse;
- it assumes all investors have access to the same information.

Some of these arose because some advocates of MPT did not understand its full generality, and stated more restrictive assumptions in their work that were later attacked by critics. The first claims above are examples of this. We saw that MPT does not assume asset returns are normally distributed, and does not assume an efficient market hypothesis. Other such claims arose because some critics of MPT did not understand it. The last two claims above are examples of this. In fact, without investor diversity it is unlikely that the IID assumptions that underpin our models would be valid.

Most modern portfolio theories are built upon more complicated models than those presented here. Many of these use mathematical tools that one sees in some graduate courses on stochastic processes. One example is *Stochastic Portfolio Theory* developed by Robert Fernholtz and others.

Finally, the simple MPT models discussed here do not consider derivatives that can be used to hedge a portfolio. These can reduce your risk by paying someone to take it on when certain contingencies are met. In other words, they are insurance policies for risky assets. They thereby transfer the risk held by individual investors to the system as a whole. This is often called *securitization of risk*. Traditional derivatives are *put* and *call* options, but since the 1980s there has been an explosion in derivative products such as exotic options, swaps, futures, and forwards. As we saw in 2008 and 2011, without proper regulation these tools can create ties that critically weaken the entire financial system.

**Thank You!**