

Solutions of Sample Problems for Second In-Class Exam
Math 246, Fall 2013, Professor David Levermore

- (1) Give the interval of definition for the solution of the initial-value problem

$$\frac{d^3x}{dt^3} + \frac{\cos(3t)}{4-t} \frac{dx}{dt} = \frac{e^{-2t}}{1+t}, \quad x(2) = x'(2) = x''(2) = 0.$$

Solution. The coefficient and forcing are both continuous over the interval $(-1, 4)$, which contains the initial time $t = 2$. The coefficient is not defined at $t = 4$ while the forcing is not defined at $t = -1$. Therefore the interval of definition is $(-1, 4)$.

- (2) Suppose that $Y_1(t)$ and $Y_2(t)$ are solutions of the differential equation

$$y'' + 2y' + (1 + t^2)y = 0.$$

Suppose you know that $W[Y_1, Y_2](0) = 5$. What is $W[Y_1, Y_2](t)$?

Solution. Abel's Theorem states that $w(t) = W[Y_1, Y_2](t)$ satisfies $w' + 2w = 0$. It follows that $w(t) = w(0)e^{-2t}$. Because $w(0) = W[Y_1, Y_2](0) = 5$, we obtain $w(t) = 5e^{-2t}$. Therefore

$$W[Y_1, Y_2](t) = 5e^{-2t}.$$

- (3) The function $Y(t) = t$ is a solution of the differential equation

$$(t^2 + 4)y'' - 2ty' + 2y = 0.$$

Find a general solution of this equation.

Solution. This is a homogeneous equation with variable coefficients. We can find another solution by order reduction. Set $y = tu$, whereby

$$y' = tu' + u, \quad y'' = tu'' + 2u'.$$

Upon substituting these expressions into the equation we obtain

$$\begin{aligned} 0 &= (t^2 + 4)y'' - 2ty' + 2y \\ &= (t^2 + 4)(tu'' + 2u') - 2t(tu' + u) + 2tu \\ &= (t^2 + 4)tu'' + (2t^2 + 8 - 2t^2)u' + (-2t + 2t)u \\ &= (t^2 + 4)tu'' + 8u'. \end{aligned}$$

This is equivalent to the homogeneous first-order equation

$$w' + \frac{8}{(t^2 + 4)t} w = 0, \quad \text{where } w = u'.$$

which has solution

$$w = \exp\left(-\int \frac{8}{(t^2 + 4)t} dt\right).$$

By the partial fraction identity

$$\frac{8}{(t^2 + 4)t} = \frac{2}{t} - \frac{2t}{t^2 + 4},$$

we see that

$$\begin{aligned}\int \frac{8}{(t^2 + 4)t} dt &= \int \frac{2}{t} - \frac{2t}{t^2 + 4} dt \\ &= \log(t^2) - \log(t^2 + 4) + c = \log\left(\frac{t^2}{t^2 + 4}\right) + c.\end{aligned}$$

Because $w = u'$ we see that

$$u' = w = c_1 \frac{t^2 + 4}{t^2} = c_1 \left(1 + \frac{4}{t^2}\right),$$

whereby

$$u = c_1 \left(t - \frac{4}{t}\right) + c_2.$$

Therefore a general solution is

$$y = tu = c_1(t^2 - 4) + c_2t.$$

- (4) Show that the functions $Y_1(t) = \cos(t)$, $Y_2(t) = \sin(t)$, and $Y_3(t) = 1$ are linearly independent.

Solution. The Wronskian of these functions is

$$\begin{aligned}W[Y_1, Y_2, Y_3](t) &= \det \begin{pmatrix} \cos(t) & \sin(t) & 1 \\ -\sin(t) & \cos(t) & 0 \\ -\cos(t) & -\sin(t) & 0 \end{pmatrix} \\ &= 1 \cdot (-\sin(t)) \cdot (-\sin(t)) - (-\cos(t)) \cdot \cos(t) \cdot 1 \\ &= \sin(t)^2 + \cos(t)^2 = 1.\end{aligned}$$

Because $W[Y_1, Y_2, Y_3](t) \neq 0$ the functions are linearly independent.

Alternative Solution. Suppose that

$$0 = c_1 Y_1(t) + c_2 Y_2(t) + c_3 Y_3(t) = c_1 \cos(t) + c_2 \sin(t) + c_3.$$

To show linear independence we must show that $c_1 = c_2 = c_3 = 0$. But setting $t = 0$, $t = \pi$, and $t = \frac{\pi}{2}$ into this relation yields the linear algebraic system

$$\begin{aligned}0 &= c_1 \cos(0) + c_2 \sin(0) + c_3 = c_1 + c_3, \\ 0 &= c_1 \cos(\pi) + c_2 \sin(\pi) + c_3 = -c_1 + c_3, \\ 0 &= c_1 \cos\left(\frac{\pi}{2}\right) + c_2 \sin\left(\frac{\pi}{2}\right) + c_3 = c_2 + c_3.\end{aligned}$$

By subtracting the second equation from the first we see that $c_1 = 0$. By adding the first two equations we see that $c_3 = 0$. By plugging $c_3 = 0$ into the third equation we see that $c_2 = 0$. Therefore $c_1 = c_2 = c_3 = 0$, whereby $Y_1(t)$, $Y_2(t)$, and $Y_3(t)$ are linearly independent.

- (5) Let L be a linear ordinary differential operator with constant coefficients. Suppose that all the roots of its characteristic polynomial (listed with their multiplicities) are $-2 + i3$, $-2 - i3$, $i7$, $i7$, $-i7$, $-i7$, 5 , 5 , 5 , -3 , 0 , 0 .

(a) Give the order of L .

Solution. There are 12 roots listed, so the degree of the characteristic polynomial is 12, whereby the order of L is 12.

(b) Give a general real solution of the homogeneous equation $Ly = 0$.

Solution. A general solution is

$$\begin{aligned} y = & c_1 e^{-2t} \cos(3t) + c_2 e^{-2t} \sin(3t) \\ & + c_3 \cos(7t) + c_4 \sin(7t) + c_5 t \cos(7t) + c_6 t \sin(7t) \\ & + c_7 e^{5t} + c_8 t e^{5t} + c_9 t^2 e^{5t} + c_{10} e^{-3t} + c_{11} + c_{12} t. \end{aligned}$$

The reasoning is as follows:

- the single conjugate pair $-2 \pm i3$ yields $e^{-2t} \cos(3t)$ and $e^{-2t} \sin(3t)$;
- the double conjugate pair $\pm i7$ yields

$$\cos(7t), \quad \sin(7t), \quad t \cos(7t), \quad \text{and} \quad t \sin(7t);$$

- the triple real root 5 yields e^{5t} , $t e^{5t}$, and $t^2 e^{5t}$;
- the single real root -3 yields e^{-3t} ;
- the double real root 0 yields 1 and t .

- (6) Give the natural fundamental set of solutions associated with $t = 0$ for each of the following equations.

(a) $y'' - 6y' + 9y = 0$.

Solution. The general initial-value problem associated with $t = 0$ is

$$y'' - 6y' + 9y = 0, \quad y(0) = y_0, \quad y'(0) = y_1.$$

This is a constant coefficient, homogeneous, linear equation. Its characteristic polynomial is

$$p(z) = z^2 - 6z + 9 = (z - 3)^2.$$

This has the double real root 3 , which yields a general solution

$$y(t) = c_1 e^{3t} + c_2 t e^{3t}.$$

Because

$$y'(t) = 3c_1 e^{3t} + 3c_2 t e^{3t} + c_2 e^{3t},$$

when the general initial conditions are imposed, we find

$$y(0) = c_1 = y_0, \quad y'(0) = 3c_1 + c_2 = y_1.$$

These are solved to find $c_1 = y_0$ and $c_2 = y_1 - 3y_0$. So the solution of the general initial-value problem is

$$y(t) = y_0 e^{3t} + (y_1 - 3y_0)t e^{3t} = (1 - 3t)e^{3t}y_0 + t e^{3t}y_1.$$

Therefore the natural fundamental set of solutions associated with $t = 0$ is

$$N_0(t) = (1 - 3t)e^{3t}, \quad N_1(t) = t e^{3t}.$$

(b) $y'' + 4y' + 20y = 0$.

Solution. The general initial-value problem associated with $t = 0$ is

$$y'' + 4y' + 20y = 0, \quad y(0) = y_0, \quad y'(0) = y_1.$$

This is a constant coefficient, homogeneous, linear equation. Its characteristic polynomial is

$$p(z) = z^2 + 4z + 20 = (z + 2)^2 + 4^2.$$

This has the conjugate pair of roots $-2 \pm 4i$, which yields a general solution

$$y(t) = c_1 e^{-2t} \cos(4t) + c_2 e^{-2t} \sin(4t).$$

Because

$$y'(t) = -2c_1 e^{-2t} \cos(4t) - 4c_1 e^{-2t} \sin(4t) - 2c_2 e^{-2t} \sin(4t) + 4c_2 e^{-2t} \cos(4t).$$

when the general initial conditions are imposed, we find

$$y(0) = c_1 = y_0, \quad y'(0) = -2c_1 + 4c_2 = y_1.$$

These are solved to find $c_1 = y_0$ and $c_2 = (y_1 + 2y_0)/4$. So the solution of the general initial-value problem is

$$\begin{aligned} y(t) &= y_0 e^{-2t} \cos(4t) + \frac{y_1 + 2y_0}{4} e^{-2t} \sin(4t) \\ &= e^{-2t} \left(\cos(4t) + \frac{1}{2} \sin(4t) \right) y_0 + e^{-2t} \frac{1}{4} \sin(4t) y_1. \end{aligned}$$

Therefore the natural fundamental set of solutions associated with $t = 0$ is

$$N_0(t) = e^{-2t} \left(\cos(4t) + \frac{1}{2} \sin(4t) \right), \quad N_1(t) = e^{-2t} \frac{1}{4} \sin(4t).$$

(7) Let $D = \frac{d}{dt}$. Solve each of the following initial-value problems.

(a) $D^2 y + 4Dy + 4y = 0, \quad y(0) = 1, \quad y'(0) = 0.$

Solution. This is a constant coefficient, homogeneous, linear equation. Its characteristic polynomial is

$$p(z) = z^2 + 4z + 4 = (z + 2)^2.$$

This has the double real root -2 , which yields a general solution

$$y(t) = c_1 e^{-2t} + c_2 t e^{-2t}.$$

Because

$$y'(t) = -2c_1 e^{-2t} - 2c_2 t e^{-2t} + c_2 e^{-2t},$$

when the initial conditions are imposed, we find that

$$y(0) = c_1 = 1, \quad y'(0) = -2c_1 + c_2 = 0.$$

These are solved to find $c_1 = 1$ and $c_2 = 2$. Therefore the solution of the initial-value problem is

$$y(t) = e^{-2t} + 2t e^{-2t} = (1 + 2t)e^{-2t}.$$

(b) $D^2y + 9y = 20e^t$, $y(0) = 0$, $y'(0) = 0$.

Solution. This is a constant coefficient, nonhomogeneous, linear equation. Its characteristic polynomial is

$$p(z) = z^2 + 9 = z^2 + 3^2.$$

This has the conjugate pair of roots $\pm i3$, which yields a general solution of the associated homogeneous problem

$$y_H(t) = c_1 \cos(3t) + c_2 \sin(3t).$$

The forcing $20e^t$ has characteristic $\mu + i\nu = 1$, degree $d = 0$, and multiplicity $m = 0$. A particular solution $y_P(t)$ can be found using either Key Identity evaluations or undetermined coefficients.

Key Identity Evaluations. Because $m + d = 0$, we can use the zero degree formula

$$L\left(\frac{e^t}{p(1)}\right) = e^t,$$

or simply evaluate the Key identity at $z = 1$, to find

$$L(e^t) = p(1)e^t = (1^2 + 9)e^t = 10e^t.$$

Multiplying this equation by 2 yields $L(2e^t) = 20e^t$. Hence, $y_P(t) = 2e^t$.

Undetermined Coefficients. Because $m = d = 0$, we seek a particular solution of the form

$$y_P(t) = Ae^t.$$

Because

$$y'_P(t) = Ae^t, \quad y''_P(t) = Ae^t,$$

we see that

$$Ly_P(t) = y''_P(t) + 9y_P(t) = Ae^t + 9Ae^t = 10Ae^t.$$

Setting $Ly_P(t) = 10Ae^t = 20e^t$, we see that $A = 2$. Hence, $y_P(t) = 2e^t$.

By either approach we find $y_P(t) = 2e^t$, which yields the general solution

$$y(t) = c_1 \cos(3t) + c_2 \sin(3t) + 2e^t.$$

Because

$$y'(t) = -3c_1 \sin(3t) + 3c_2 \cos(3t) + 2e^t,$$

when the initial conditions are imposed we find that

$$y(0) = c_1 + 2 = 0, \quad y'(0) = 3c_2 + 2 = 0.$$

These are solved to obtain $c_1 = -2$ and $c_2 = -\frac{2}{3}$. Therefore the solution of the initial-value problem is

$$y(t) = -2 \cos(3t) - \frac{2}{3} \sin(3t) + 2e^t.$$

(8) Let $D = \frac{d}{dt}$. Give a general real solution for each of the following equations.

(a) $D^2y + 4Dy + 5y = 3\cos(2t)$.

Solution. This is a constant coefficient, nonhomogeneous, linear equation. Its characteristic polynomial is

$$p(z) = z^2 + 4z + 5 = (z + 2)^2 + 1.$$

This has the conjugate pair of roots $-2 \pm i$, which yields a general solution of the associated homogeneous problem

$$y_H(t) = c_1 e^{-2t} \cos(t) + c_2 e^{-2t} \sin(t).$$

The forcing $3\cos(2t)$ has characteristic $\mu + i\nu = i2$, degree $d = 0$, and multiplicity $m = 0$. A particular solution $y_P(t)$ can be found using either Key Identity evaluations or undetermined coefficients.

Key Identity Evaluations. Because $m + d = 0$, we can use the zero degree formula

$$L\left(\frac{e^{i2t}}{p(i2)}\right) = e^{i2t},$$

or simply evaluate the Key identity at $z = i2$, to find

$$L(e^{i2t}) = p(i2)e^{i2t} = ((i2)^2 + 4(i2) + 5)e^{i2t} = (1 + i8)e^{i2t}.$$

Because the forcing $3\cos(2t) = 3\operatorname{Re}(e^{i2t})$, we divide the above by $1 + i8$ and multiply by 3 to find

$$L\left(\frac{3}{1 + i8} e^{i2t}\right) = 3e^{i2t}.$$

Hence,

$$\begin{aligned} y_P(t) &= \operatorname{Re}\left(\frac{3}{1 + i8} e^{i2t}\right) = \operatorname{Re}\left(\frac{3(1 - i8)}{1^2 + 8^2} e^{i2t}\right) = \frac{3}{65} \operatorname{Re}((1 - i8)e^{i2t}) \\ &= \frac{3}{65} (\cos(2t) + 8\sin(2t)) = \frac{3}{65} \cos(2t) + \frac{24}{65} \sin(2t). \end{aligned}$$

Therefore a general solution is

$$y = c_1 e^{-2t} \cos(t) + c_2 e^{-2t} \sin(t) + \frac{3}{65} \cos(2t) + \frac{24}{65} \sin(2t).$$

Undetermined Coefficients. Because $m = d = 0$, we seek a particular solution of the form

$$y_P(t) = A \cos(2t) + B \sin(2t).$$

Because

$$\begin{aligned} y'_P(t) &= -2A \sin(2t) + 2B \cos(2t), \\ y''_P(t) &= -4A \cos(2t) - 4B \sin(2t), \end{aligned}$$

we see that

$$\begin{aligned} \text{Ly}_P(t) &= y_P''(t) + 4y_P'(t) + 5y_P(t) \\ &= [-4A \cos(2t) - 4B \sin(2t)] + 4[-2A \sin(2t) + 2B \cos(2t)] \\ &\quad + 5[A \cos(2t) + B \sin(2t)] \\ &= (A + 8B) \cos(2t) + (B - 8A) \sin(2t). \end{aligned}$$

Setting $\text{Ly}_P(t) = (A + 8B) \cos(2t) + (B - 8A) \sin(2t) = 3 \cos(2t)$, we see that

$$A + 8B = 3, \quad B - 8A = 0.$$

We find that $A = \frac{3}{65}$ and $B = \frac{24}{65}$. Hence, $y_P(t) = \frac{3}{65} \cos(2t) + \frac{24}{65} \sin(2t)$. Therefore a general solution is

$$y = c_1 e^{-2t} \cos(t) + c_2 e^{-2t} \sin(t) + \frac{3}{65} \cos(2t) + \frac{24}{65} \sin(2t).$$

(b) $D^2 y - y = t e^t$.

Solution. This is a constant coefficient, nonhomogeneous, linear equation. Its characteristic polynomial is

$$p(z) = z^2 - 1 = (z + 1)(z - 1).$$

This has the real roots -1 and 1 , which yields a general solution of the associated homogeneous problem

$$y_H(t) = c_1 e^{-t} + c_2 e^t.$$

The forcing $t e^t$ has characteristic $\mu + i\nu = 1$, degree $d = 1$, and multiplicity $m = 1$. A particular solution $y_P(t)$ can be found using either Key Identity evaluations or undetermined coefficients.

Key Identity Evaluations. Because $m + d = 2$, we need the Key identity and its first two derivatives

$$\begin{aligned} L(e^{zt}) &= (z^2 - 1)e^{zt}, \\ L(t e^{zt}) &= (z^2 - 1)t e^{zt} + 2z e^{zt}, \\ L(t^2 e^{zt}) &= (z^2 - 1)t^2 e^{zt} + 4zt e^{zt} + 2e^{zt}. \end{aligned}$$

Evaluate these at $z = 1$ to find

$$L(e^t) = 0, \quad L(t e^t) = 2e^t, \quad L(t^2 e^t) = 4t e^t + 2e^t.$$

Subtracting the second equation from the third yields

$$L(t^2 e^t - t e^t) = 4t e^t.$$

Dividing this equation by 4 gives $L(\frac{1}{4}(t^2 - t) e^t) = t e^t$. Hence, $y_P(t) = \frac{1}{4}(t^2 - t) e^t$. Therefore a general solution is

$$y = c_1 e^{-t} + c_2 e^t + \frac{1}{4}(t^2 - t) e^t.$$

Undetermined Coefficients. Because $m = 1$ and $d = 1$, we seek a particular solution of the form

$$y_P(t) = (A_0 t^2 + A_1 t) e^t,$$

Because

$$\begin{aligned} y'_P(t) &= (A_0 t^2 + A_1 t) e^t + (2A_0 t + A_1) e^t \\ &= (A_0 t^2 + (2A_0 + A_1)t + A_1) e^t, \\ y''_P(t) &= (A_0 t^2 + (2A_0 + A_1)t + A_1) e^t + (2A_0 t + (2A_0 + A_1)) e^t \\ &= (A_0 t^2 + (4A_0 + A_1)t + 2A_0 + 2A_1) e^t, \end{aligned}$$

we see that

$$\begin{aligned} \text{Ly}_P(t) &= y''_P(t) - y_P(t) \\ &= (A_0 t^2 + (4A_0 + A_1)t + 2A_0 + 2A_1) e^t - (A_0 t^2 + A_1 t) e^t \\ &= (4A_0 t + 2A_0 + 2A_1) e^t = 4A_0 t e^t + 2(A_0 + A_1) e^t. \end{aligned}$$

Setting $\text{Ly}_P(t) = 4A_0 t e^t + 2(A_0 + A_1) e^t = t e^t$, we obtain $4A_0 = 1$ and $A_0 + A_1 = 0$. It follows that $A_0 = \frac{1}{4}$ and $A_1 = -\frac{1}{4}$. Hence, $y_P(t) = \frac{1}{4}(t^2 - t) e^t$. Therefore a general solution is

$$y = c_1 e^{-t} + c_2 e^t + \frac{1}{4}(t^2 - t) e^t.$$

(c) $\text{D}^2 y - y = \frac{1}{1 + e^t}.$

Solution. This is a constant coefficient, nonhomogeneous, linear equation. Its characteristic polynomial is

$$p(z) = z^2 - 1 = (z - 1)(z + 1).$$

This has the real roots 1 and -1 , which yields a general solution of the associated homogeneous problem

$$y_H(t) = c_1 e^t + c_2 e^{-t}.$$

The forcing does not have the form needed for undetermined coefficients. Therefore we must use either the Green function method or the variation of parameters method.

Green Function. The Green function $g(t)$ satisfies

$$\text{D}^2 g - g = 0, \quad g(0) = 0, \quad g'(0) = 1.$$

Set $g(t) = c_1 e^t + c_2 e^{-t}$. The first initial condition implies $g(0) = c_1 + c_2 = 0$. Because $g'(t) = c_1 e^t - c_2 e^{-t}$, the second initial condition yields $g'(0) = c_1 - c_2 = 1$. It follows that $c_1 = \frac{1}{2}$ and $c_2 = -\frac{1}{2}$, whereby $g(t) = \frac{1}{2}(e^t - e^{-t})$. Hence, a particular solution is

$$\begin{aligned} Y_P(t) &= \frac{1}{2} \int_0^t (e^{t-s} - e^{-t+s}) \frac{1}{1 + e^s} ds \\ &= \frac{1}{2} e^t \int_0^t \frac{e^{-s}}{1 + e^s} ds - \frac{1}{2} e^{-t} \int_0^t \frac{e^s}{1 + e^s} ds. \end{aligned}$$

The definite integrals on the right-hand side can be evaluated as

$$\begin{aligned}\int_0^t \frac{e^{-s}}{1+e^s} ds &= \int_0^t \frac{e^{-2s}}{e^{-s}+1} ds = \int_0^t e^{-s} - \frac{e^{-s}}{e^{-s}+1} ds \\ &= \left[-e^{-s} + \log(e^{-s}+1) \right] \Big|_{s=0}^t = 1 - e^{-t} + \log\left(\frac{e^{-t}+1}{2}\right), \\ \int_0^t \frac{e^s}{1+e^s} ds &= \log(1+e^s) \Big|_{s=0}^t = \log\left(\frac{1+e^t}{2}\right).\end{aligned}$$

Hence, the particular solution $Y_P(t)$ is given by

$$Y_P(t) = \frac{1}{2} \left[e^t - 1 + e^t \log\left(\frac{e^{-t}+1}{2}\right) \right] - \frac{1}{2} e^{-t} \log\left(\frac{1+e^t}{2}\right).$$

Therefore a general solution is $y = Y_H(t) + Y_P(t)$ where $Y_H(t)$ and $Y_P(t)$ are given above. This yields

$$y = c_1 e^t + c_2 e^{-t} + \frac{1}{2} \left[e^t - 1 + e^t \log\left(\frac{e^{-t}+1}{2}\right) \right] - \frac{1}{2} e^{-t} \log\left(\frac{1+e^t}{2}\right).$$

Variation of Parameters. Seek a solution in the form

$$y = u_1(t) e^t + u_2(t) e^{-t},$$

where $u_1(t)$ and $u_2(t)$ satisfy

$$\begin{aligned}u_1'(t) e^t + u_2'(t) e^{-t} &= 0, \\ u_1'(t) e^t - u_2'(t) e^{-t} &= \frac{1}{1+e^t}.\end{aligned}$$

Solve this system to obtain

$$u_1'(t) = \frac{1}{2} \frac{e^{-t}}{1+e^t}, \quad u_2'(t) = -\frac{1}{2} \frac{e^t}{1+e^t}.$$

Integrate these equations to find

$$\begin{aligned}u_1(t) &= \frac{1}{2} \int \frac{e^{-t}}{1+e^t} dt = \frac{1}{2} \int \frac{e^{-2t}}{e^{-t}+1} dt \\ &= \frac{1}{2} \int e^{-t} - \frac{e^{-t}}{e^{-t}+1} dt = -\frac{1}{2} e^{-t} + \frac{1}{2} \log(e^{-t}+1) + c_1, \\ u_2(t) &= -\frac{1}{2} \int \frac{e^t}{1+e^t} dt = -\frac{1}{2} \log(1+e^t) + c_2.\end{aligned}$$

Therefore a general solution is

$$y = c_1 e^t + c_2 e^{-t} - \frac{1}{2} + \frac{1}{2} e^t \log(e^{-t}+1) - \frac{1}{2} e^{-t} \log(1+e^t).$$

- (9) Let $D = \frac{d}{dt}$. Consider the equation

$$Ly = D^2y - 6Dy + 25y = e^{t^2}.$$

- (a) Compute the Green function $g(t)$ associated with L .

Solution. The Green function $g(t)$ satisfies

$$D^2g - 6Dg + 25g = 0, \quad g(0) = 0, \quad g'(0) = 1.$$

The characteristic polynomial of L is $p(z) = z^2 - 6z + 25 = (z - 3)^2 + 4^2$, which has roots $3 \pm i4$. Set $g(t) = c_1 e^{3t} \cos(4t) + c_2 e^{3t} \sin(4t)$. The first initial condition implies $g(0) = c_1 = 0$, whereby $g(t) = c_2 e^{3t} \sin(4t)$. Because $g'(t) = 3c_2 e^{3t} \sin(4t) + 4c_2 e^{3t} \cos(4t)$, the second initial condition implies $g'(0) = 4c_2 = 1$, whereby $c_2 = \frac{1}{4}$. Therefore the Green function associated with L is given by

$$g(t) = \frac{1}{4} e^{3t} \sin(4t).$$

- (b) Use the Green function to express a particular solution $Y_P(t)$ in terms of definite integrals.

Solution. A particular solution $Y_P(t)$ is given by

$$Y_P(t) = \int_0^t g(t-s) e^{s^2} ds = \frac{1}{4} \int_0^t e^{3(t-s)} \sin(4(t-s)) e^{s^2} ds.$$

Because $\sin(4(t-s)) = \sin(4t) \cos(4s) - \cos(4t) \sin(4s)$, this particular solution is given in terms of definite integrals as

$$Y_P(t) = \frac{1}{4} e^{3t} \sin(4t) \int_0^t e^{-3s} \cos(4s) e^{s^2} ds - \frac{1}{4} e^{3t} \cos(4t) \int_0^t e^{-3s} \sin(4s) e^{s^2} ds.$$

Remark. The above definite integrals cannot be evaluated analytically.

- (10) The functions t and t^2 are solutions of the homogeneous equation

$$t^2 \frac{d^2 y}{dt^2} - 2t \frac{dy}{dt} + 2y = 0 \quad \text{over } t > 0.$$

(You do not have to check that this is true!)

- (a) Compute their Wronskian.

Solution. The Wronskian is

$$W[t, t^2](t) = \det \begin{pmatrix} t & t^2 \\ 1 & 2t \end{pmatrix} = t \cdot (2t) - 1 \cdot t^2 = 2t^2 - t^2 = t^2.$$

- (b) Solve the initial-value problem

$$t^2 \frac{d^2 y}{dt^2} - 2t \frac{dy}{dt} + 2y = t^3 e^t, \quad y(1) = y'(1) = 0, \quad \text{over } t > 0.$$

Try to evaluate all definite integrals explicitly.

Solution. Because this problem has variable coefficients, we must use either the general Green function method or the variation of parameters method to solve it. To apply either method we must first bring the equation into its normal form

$$\frac{d^2y}{dt^2} - \frac{2}{t} \frac{dy}{dt} + \frac{2}{t^2} y = te^t \quad \text{over } t > 0.$$

Because $W[t, t^2](t) = t^2 \neq 0$ over $t > 0$, we know that t and t^2 constitute a fundamental set of solutions to the associated homogeneous equation.

General Green Function. The Green function $G(t, s)$ is given by

$$G(t, s) = \frac{\det \begin{pmatrix} s & s^2 \\ t & t^2 \end{pmatrix}}{\det \begin{pmatrix} s & s^2 \\ 1 & 2s \end{pmatrix}} = \frac{st^2 - ts^2}{2s^2 - s^2} = (t - s) \frac{t}{s}.$$

The Green function formula then yields the solution

$$\begin{aligned} y(t) &= \int_1^t G(t, s) se^s ds = t \int_1^t (t - s)e^s ds = t^2 \int_1^t e^s ds - t \int_1^t se^s ds \\ &= t^2(e^t - e) - t(t - 1)e^t = -et^2 + te^t. \end{aligned}$$

Variation of Parameters. A general solution of the associated homogeneous problem is

$$y_H(t) = c_1 t + c_2 t^2.$$

Seek a solution in the form

$$y = u_1(t)t + u_2(t)t^2,$$

where $u_1'(t)$ and $u_2'(t)$ satisfy

$$\begin{aligned} u_1'(t)t + u_2'(t)t^2 &= 0, \\ u_1'(t)1 + u_2'(t)2t &= te^t. \end{aligned}$$

Solve this system to obtain

$$u_1'(t) = -te^t, \quad u_2'(t) = e^t.$$

Integrate these equations to find

$$u_1(t) = c_1 + (1 - t)e^t, \quad u_2(t) = c_2 + e^t.$$

Therefore a general solution is

$$y(t) = c_1 t + c_2 t^2 + (1 - t)e^t t + e^t t^2 = c_1 t + c_2 t^2 + te^t.$$

Because

$$y'(t) = c_1 + 2c_2 t + (t + 1)e^t,$$

when the initial conditions are imposed we find that

$$y(1) = c_1 + c_2 + e = 0, \quad y'(1) = c_1 + 2c_2 + 2e = 0.$$

These are solved to obtain $c_1 = 0$ and $c_2 = -e$. Therefore the solution of the initial-value problem is

$$y(t) = -et^2 + te^t.$$

(11) What answer will be produced by the following MATLAB commands?

```
>> ode1 = 'D2y + 2*Dy + 5*y = 16*exp(t)';
>> dsolve(ode1, 't')
ans =
```

Solution. The commands ask MATLAB to give the general solution of the equation

$$D^2y + 2Dy + 5y = 16e^t, \quad \text{where } D = \frac{d}{dt}.$$

MATLAB will produce the answer

$$2*\exp(t) + C1*\exp(-t)*\sin(2*t) + C2*\exp(-t)*\cos(2*t)$$

This can be seen as follows. This is a constant coefficient, nonhomogeneous, linear equation. The characteristic polynomial is

$$p(z) = z^2 + 2z + 5 = (z + 1)^2 + 4 = (z + 1)^2 + 2^2.$$

Its roots are the conjugate pair $-1 \pm i2$. A general solution of the associated homogeneous problem is

$$y_H(t) = c_1 e^{-t} \cos(2t) + c_2 e^{-t} \sin(2t).$$

The forcing $16e^t$ has characteristic $\mu + i\nu = 1$, degree $d = 0$, and multiplicity $m = 0$. A particular solution $y_P(t)$ can be found using either Key Identity evaluations or undetermined coefficients.

Key Identity Evaluations. Because $m + d = 0$, we only need to evaluate the Key identity at $z = 1$, to find

$$L(e^t) = p(1)e^t = (1^2 + 2 \cdot 1 + 5)e^t = 8e^t.$$

Multiply this by 2 to obtain $L(2e^t) = 16e^t$. Hence, $y_P(t) = 2e^t$.

Undetermined Coefficients. Because $m = d = 0$, we seek a particular solution of the form

$$y_P(t) = Ae^t.$$

Because

$$y'_P(t) = Ae^t, \quad y''_P(t) = Ae^t,$$

we see that

$$Ly_P(t) = y''_P(t) + 2y'_P(t) + 5y_P(t) = [Ae^t] + 2[Ae^t] + 5[Ae^t] = 8Ae^t.$$

Setting $Ly_P(t) = 8Ae^t = 16e^t$, we see that $A = 2$. Hence, $y_P(t) = 2e^t$.

By either approach we find $y_P(t) = 2e^t$. Therefore a general solution is

$$y = c_1 e^{-t} \cos(2t) + c_2 e^{-t} \sin(2t) + 2e^t.$$

Up to notational differences, this is the answer that MATLAB produces.

(12) The vertical displacement of a mass on a spring is given by

$$h(t) = 4e^{-t} \cos(7t) - 3e^{-t} \sin(7t),$$

where positive displacements are upward.

- (a) Express $h(t)$ in the form $h(t) = Ae^{-t} \cos(\omega t - \delta)$ with $A > 0$ and $0 \leq \delta < 2\pi$, identifying the quasiperiod and phase of the oscillation. (The phase may be expressed in terms of an inverse trig function.)

Solution. By comparing

$$Ae^{-t} \cos(\omega t - \delta) = Ae^{-t} \cos(\delta) \cos(\omega t) + Ae^{-t} \sin(\delta) \sin(\omega t),$$

with $h(t) = 4e^{-t} \cos(7t) - 3e^{-t} \sin(7t)$, we see that $\omega = 7$ and that

$$A \cos(\delta) = 4, \quad A \sin(\delta) = -3.$$

This shows that (A, δ) are the polar coordinates of the point in the plane whose Cartesian coordinates are $(4, -3)$. Clearly A is given by

$$A = \sqrt{4^2 + 3^2} = \sqrt{16 + 9} = \sqrt{25} = 5.$$

Because $(4, -3)$ lies in the fourth quadrant, the phase δ satisfies $\frac{3\pi}{2} < \delta < 2\pi$. There are several ways to express δ . A picture shows that if we use 2π as a reference then

$$\sin(2\pi - \delta) = \frac{3}{5}, \quad \tan(2\pi - \delta) = \frac{3}{4}, \quad \cos(2\pi - \delta) = \frac{4}{5},$$

and we can express the phase by any one of the formulas

$$\delta = 2\pi - \sin^{-1}\left(\frac{3}{5}\right), \quad \delta = 2\pi - \tan^{-1}\left(\frac{3}{4}\right), \quad \delta = 2\pi - \cos^{-1}\left(\frac{4}{5}\right).$$

Finally, because the quasifrequency is $\omega = 7$, the quasiperiod T is given by

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{7}.$$

- (b) Sketch the solution over $0 \leq t \leq 2$.

Solution. This will be shown during the review session provided someone asks for it.

(13) When a mass of 4 grams is hung vertically from a spring, at rest it stretches the spring 9.8 cm. (Gravitational acceleration is $g = 980 \text{ cm/sec}^2$.) At $t = 0$ the mass is displaced 3 cm above its equilibrium position and is released with no initial velocity. It moves in a medium that imparts a drag force of 2 dynes ($1 \text{ dyne} = 1 \text{ gram cm/sec}^2$) when the speed of the mass is 4 cm/sec. There are no other forces. (Assume that the spring force is proportional to displacement and that the drag force is proportional to velocity.)

- (a) Formulate an initial-value problem that governs the motion of the mass for $t > 0$. (DO NOT solve this initial-value problem, just write it down!)

Solution. Let $h(t)$ be the displacement of the mass from its equilibrium (rest) position at time t in centimeters, with upward displacements being positive. The governing initial-value problem then has the form

$$m \frac{d^2 h}{dt^2} + \gamma \frac{dh}{dt} + kh = 0, \quad h(0) = 3, \quad h'(0) = 0,$$

where m is the mass, γ is the drag coefficient, and k is the spring constant. The problem says that $m = 4$ grams. The spring constant is obtained by balancing the weight of the mass ($mg = 4 \cdot 980$ dynes) with the force applied by the spring when it is stretched 9.8 cm. This gives $k \cdot 9.8 = 4 \cdot 980$, or

$$k = \frac{4 \cdot 980}{9.8} = 400 \quad \text{dynes/cm}.$$

The drag coefficient is obtained by balancing the force of 2 dynes with the drag force imparted by the medium when the speed of the mass is 4 cm/sec. This gives $\gamma \cdot 4 = 2$, or

$$\gamma = \frac{2}{4} = \frac{1}{2} \quad \text{dynes sec/cm}.$$

Therefore the governing initial-value problem is

$$4 \frac{d^2 h}{dt^2} + \frac{1}{2} \frac{dh}{dt} + 400h = 0, \quad h(0) = 3, \quad h'(0) = 0,$$

If we had chosen downward displacements to be positive then the governing initial-value problem would be the same except for the first initial condition, which would then be $h(0) = -3$.

- (b) What is the natural frequency of the spring?

Solution. The natural frequency of the spring is given by

$$\omega_o = \sqrt{\frac{k}{m}} = \sqrt{\frac{4 \cdot 980}{4 \cdot 9.8}} = \sqrt{100} = 10 \quad 1/\text{sec}.$$

- (c) Show that the system is under damped and find its quasifrequency.

Solution. The characteristic polynomial is

$$p(z) = z^2 + \frac{1}{8}z + 100 = \left(z + \frac{1}{16}\right)^2 + 100 - \frac{1}{16^2},$$

which has a conjugate pair of roots. Therefore the system is under damped. The roots are $-\frac{1}{16} \pm i\nu$ where

$$\nu = \sqrt{100 - \frac{1}{16^2}} \quad 1/\text{sec}.$$

This is the quasifrequency.