UCLA Math 135, Winter 2015
Ordinary Differential Equations

5. Nonhomogeneous Linear Equations with Constant Coefficients

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January 5, 2015

Contents

Return to Main Webpage

5. Nonhomogeneous Linear Equations with Constant Coefficients

This chapter gives three methods by which we can construct particular solutions to an n^{th}-order nonhomogeneous linear ordinary differential equation

$$\mathrm{L}y = f(t)\,, \tag{5.1}$$

when the differential operator L has *constant coefficients* and is in normal form,

$$\mathrm{L} = \mathrm{D}^n + a_1\mathrm{D}^{n-1} + \cdots + a_{n-1}\mathrm{D} + a_n\,. \tag{5.2}$$

The previous chapter showed that constructing a particular solution of (5.1) is the key step either in finding a general solution of (5.1) or in solving an initial-value problem associated with (5.1).

The first two methods we present are *Key Identity Evaluations* and *Undetermined Coefficients*. They are related and require the forcing $f(t)$ to have a special form. Because they generally provide the fastest way to find a particular solution whenever $f(t)$ has this special form, it is a good idea to master at least one of them.

The third method we present is *Green Functions*. It can be applied to any forcing $f(t)$, but does not yield an explicit particular solution. Rather, it reduces the problem of computing a particular solution to that of evaluating n integrals. Because evaluating integrals takes time, this method should only be applied when the first two methods cannot be applied.

5.1. Key Identity Evaluations.

This method should only be used to find a particular solution of equation (5.1) when the following two conditions are met.

(1) The differential operator L has constant coefficients.
(2) The forcing $f(t)$ can be expressed in the form

$$\begin{aligned} f(t) = &\big(f_0t^d + f_1t^{d-1} + \cdots + f_d\big)e^{\mu t}\cos(\nu t) \\ &+ \big(g_0t^d + g_1t^{d-1} + \cdots + g_d\big)e^{\mu t}\sin(\nu t)\,, \end{aligned} \tag{5.3}$$

for some nonnegative integer d and real numbers μ and ν with $\nu \geq 0$. Here we are assuming that $f_0, f_1, \cdots, f_d$ and $g_0, g_1, \cdots, g_d$ are all real and that either $f_0 \neq 0$ or $\nu g_0 \neq 0$.

When the forcing $f(t)$ can be expressed the form (5.3) it is said to have *characteristic form*. The complex number $\mu + i\nu$ with $\nu \geq 0$ is called the *characteristic* of $f(t)$ while the integer d is called the *degree* of $f(t)$. If $\nu = 0$ we say that $f(t)$ has *real characteristic form*. Otherwise we say that $f(t)$ has *complex characteristic form*.

Remark. If $f(t)$ has real characteristic form then the g_k can be anything because $\nu = 0$ implies that $\sin(\nu t) = 0$ for every t. In this case we can assume that all the g_k are zero. Then the f_k are uniquely determined by the linear independence of the functions

$$t^de^{\mu t}\,, \qquad t^{d-1}e^{\mu t}\,, \quad \cdots \quad t\,e^{\mu t}\,, \qquad e^{\mu t}\,.$$

Remark. If $f(t)$ has complex characteristic form then we can restrict to $\nu > 0$ because the form (5.3) remains unchanged if we replace ν with $-\nu$ and each g_k with $-g_k$. The restriction $\nu > 0$ implies that the f_k and g_k are uniquely determined by the linear independence of the functions

$$\begin{array}{cccccc} t^d e^{\mu t}\cos(\nu t)\,, & t^{d-1}e^{\mu t}\cos(\nu t)\,, & \cdots & t\, e^{\mu t}\cos(\nu t)\,, & e^{\mu t}\cos(\nu t)\,, \\ t^d e^{\mu t}\sin(\nu t)\,, & t^{d-1}e^{\mu t}\sin(\nu t)\,, & \cdots & t\, e^{\mu t}\sin(\nu t)\,, & e^{\mu t}\sin(\nu t)\,. \end{array}$$

Verification of condition (1) is always easy to do by inspection. Verification of condition (2) is often easy to do by inspection, but sometimes requires the use of some trigonometric or other identity. You should be able to identify when a forcing $f(t)$ has characteristic form and, when it does, to read-off its characteristic and degree.

Example. The forcing of the differential equation $\mathrm{L}y = 2e^{2t}$ has characteristic form with characteristic $\mu + i\nu = 2$ and degree $d = 0$.

Example. The forcing of the differential equation $\mathrm{L}y = t^2 e^{-3t}$ has characteristic form with characteristic $\mu + i\nu = -3$ and degree $d = 2$.

Example. The forcing of the differential equation $\mathrm{L}y = t\, e^{5t}\sin(3t)$ has characteristic form with characteristic $\mu + i\nu = 5 + i3$ and degree $d = 1$.

Example. The forcing of the differential equation $\mathrm{L}y = t^3 + 7t$ has characteristic form with characteristic $\mu + i\nu = 0$ and degree $d = 3$.

Example. The forcing of the differential equation $\mathrm{L}y = \sin(2t)\cos(2t)$ can be put into the form (5.3) by using the double-angle identity $\sin(4t) = 2\sin(2t)\cos(2t)$. The equation thereby can be expressed as $\mathrm{L}y = \frac{1}{2}\sin(4t)$. Therefore the forcing has characteristic form with characteristic $\mu + i\nu = i4$ and degree $d = 0$.

5.1.1. *Setting Up Key Identity Evaluations.* The method of Key Identity Evaluations is based on the observation that for any forcing that has characteristic form with characteristic $\mu + i\nu$ we can construct an explicit particular solution of equation (5.1) by evaluating the Key Identity and some of its derivatives with respect to z at $z = \mu + i\nu$. For example, if $p(z)$ is the characteristic polynomial of L then the Key Identity and its first four derivatives with respect to z are

$$\begin{aligned} \mathrm{L}\big(e^{zt}\big) &= p(z)e^{zt}\,, \\ \mathrm{L}\big(t\, e^{zt}\big) &= p(z)\, t\, e^{zt} + p'(z)\, e^{zt}\,, \\ \mathrm{L}\big(t^2 e^{zt}\big) &= p(z)\, t^2 e^{zt} + 2p'(z)\, t\, e^{zt} + p''(z)\, e^{zt}\,, \\ \mathrm{L}\big(t^3 e^{zt}\big) &= p(z)\, t^3 e^{zt} + 3p'(z)\, t^2 e^{zt} + 3p''(z)\, t\, e^{zt} + p'''(z)\, e^{zt}\,, \\ \mathrm{L}\big(t^4 e^{zt}\big) &= p(z)\, t^4 e^{zt} + 4p'(z)\, t^3 e^{zt} + 6p''(z)\, t^2 e^{zt} + 4p'''(z)\, t\, e^{zt} + p^{(4)}(z)\, e^{zt}\,. \end{aligned} \tag{5.4}$$

We see that when these are evaluated at $z = \mu + i\nu$ then the terms on the right-hand sides above have the same form as those appearing in the forcing (5.3).

Remark. The coefficients appearing on the right-hand side of (5.4) are the coefficients that appear in the binomial expansion. They are generated by the Pascal triangle:

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \\
 & & & & & & \vdots & & & & & &
\end{array}
$$

Recall that each entry in the Pascal triangle is the sum of the two entries appearing immediately above it. By using this rule it is easy to extend (5.4) to as many derivatives of the Key Identity with respect to z as needed.

If the characteristic $\mu + i\nu$ is not a root of $p(z)$ then one needs through the d^{th} derivative of the Key Identity with respect to z. These should be evaluated at $z = \mu + i\nu$. A linear combination of the resulting $d + 1$ equations (and their conjugates if $\nu > 0$) can then be found so that its right-hand side equals any $f(t)$ given by (5.3). We can then read off Y_P from this linear combination.

More generally, if the characteristic $\mu + i\nu$ is a root of $p(z)$ of multiplicity m then one needs through the $(m+d)^{\text{th}}$ derivative of the Key Identity with respect to z. These should be evaluated at $z = \mu + i\nu$. Because $\mu + i\nu$ is a root of multiplicity m, the first m of these will vanish when evaluated at $z = \mu + i\nu$. A linear combination of the resulting $d + 1$ equations (and their conjugates if $\nu > 0$) can then be found so that its right-hand side equals any $f(t)$ given by (5.3). We can then read off Y_P from this linear combination. This case includes the previous one if we understand "$\mu + i\nu$ is a root of $p(z)$ of multiplicity 0" to mean that it is not a root of $p(z)$.

Given a nonhomogeneous problem $\mathrm{L}y = f(t)$ in which the forcing $f(t)$ has characteristic form with characteristic $\mu + i\nu$, degree d, and multiplicity m, the method of Key Identity Evaluations will find a particular solution Y_P as follows.

1. Write down the Key Identity through its $(m+d)^{\text{th}}$ derivative with respect to z.
2. Evaluate the m^{th} through the $(m+d)^{\text{th}}$ derivative of the Key Identity at $z = \mu + i\nu$.
3. Find a linear combination of the resulting $d+1$ equations (and their conjugates if $\nu > 0$) whose right-hand side equals $f(t)$ and read off Y_P.

Remark. The method Key Identity Evaluations is fairly painless when d is small. For the problems we will face both m and d will be small, so $m + d$ will seldom be larger than 3, and more commonly be 0, 1, or 2.

5.1.2. *Zero Degree Examples.* The case where $d = 0$ often arises in applications. When $d = 0$ the method of Key Identity evaluations reduces to a single equation that can be easily solved. For example, if the characteristic $\mu + i\nu$ has multiplicity $m = 0$ then that single equation is just the Key Identity evaluated at $\mu + i\nu$, which is

$$\mathrm{L}\big(e^{(\mu+i\nu)t}\big) = p(\mu + i\nu)\, e^{(\mu+i\nu)t} .$$

Because $p(\mu + i\nu) \neq 0$ we see that a particular solution of $\mathrm{L}(y) = e^{(\mu+i\nu)t}$ is

$$Y_P(t) = \frac{e^{(\mu+i\nu)t}}{p(\mu + i\nu)} . \tag{5.5a}$$

On the other hand, if $\mu + i\nu$ has multiplicity $m > 0$ then that single equation is just the m^{th} derivative of the Key Identity evaluated at $\mu + i\nu$, which because

$$p^{(k)}(\mu + i\nu) = 0 \quad \text{for every } k < m \text{ and } \quad p^{(m)}(\mu + i\nu) \neq 0 ,$$

we see from (5.4) becomes

$$\mathrm{L}\big(t^m e^{(\mu+i\nu)t}\big) = p^{(m)}(\mu + i\nu)\, e^{(\mu+i\nu)t} .$$

Because $p^{(m)}(\mu + i\nu) \neq 0$ we see that a particular solution of $\mathrm{L}(y) = e^{(\mu+i\nu)t}$ is

$$Y_P(t) = \frac{t^m e^{(\mu+i\nu)t}}{p^{(m)}(\mu + i\nu)} . \tag{5.5b}$$

We call (5.5) the *zero degree formulas*. The first can be recovered from the second by setting $m = 0$. They can be applied whenever the forcing has characteristic form with degree $d = 0$ and characteristic $\mu + i\nu$. We illustrate with examples.

Remark. Because the characterisitc $\mu + i\nu$ has multplicity $m > 0$ we see from (5.4) that the right-hand sides of the k^{th} derivative of the Key Identity will vanish at $z = \mu + i\nu$ for every $k < m$, which tells us something we already know, namely, that $\mathrm{L}(t^k e^{(\mu+i\nu)t}) = 0$ for $k = 0, \cdots, m - 1$. This is why we need the m^{th} derivative of the Key Identity to construct a particular solution.

Remark. The zero degree formulas (5.5) are also called *exponential response formulas* because they give a solution of equations that have a purely exponential forcing — i.e. equations in the form $\mathrm{L}y = e^{(\mu+i\nu)t}$.

In the first examples the forcing has real characteristic form, i.e. $\nu = 0$.

Example. Find a general solution of

$$\mathrm{L}y = \mathrm{D}^2 y + 2\mathrm{D}y + 10y = 6e^{2t} .$$

Solution. The characteristic polynomial is

$$p(z) = z^2 + 2z + 10 = (z + 1)^2 + 9 = (z + 1)^2 + 3^2 .$$

Its roots are $-1 \pm i3$. Hence, a general solution of the associated homogeneous equation is

$$Y_H(t) = c_1 e^{-t} \cos(3t) + c_2 e^{-t} \sin(3t) .$$

To find a particular solution, first notice that the forcing has characteristic form with characteristic $\mu + i\nu = 2$ and degree $d = 0$. Because the characteristic 2 is not a root of $p(z)$, it has multiplicity $m = 0$.

For $\mu + i\nu = 2$ and $m = 0$ the zero degree formula (5.5a) yields

$$\mathrm{L}\left(\frac{e^{2t}}{p(2)}\right) = e^{2t}\,.$$

By multiplying this equation by 6 we see that a particular solution of $\mathrm{L}(y) = 6e^{2t}$ is

$$y_P(t) = 6\,\frac{e^{2t}}{p(2)} = 6\,\frac{e^{2t}}{18} = \tfrac{1}{3}e^{2t}\,.$$

Therefore a general solution is

$$y = c_1 e^{-t}\cos(3t) + c_2 e^{-t}\sin(3t) + \tfrac{1}{3}e^{2t}\,.$$

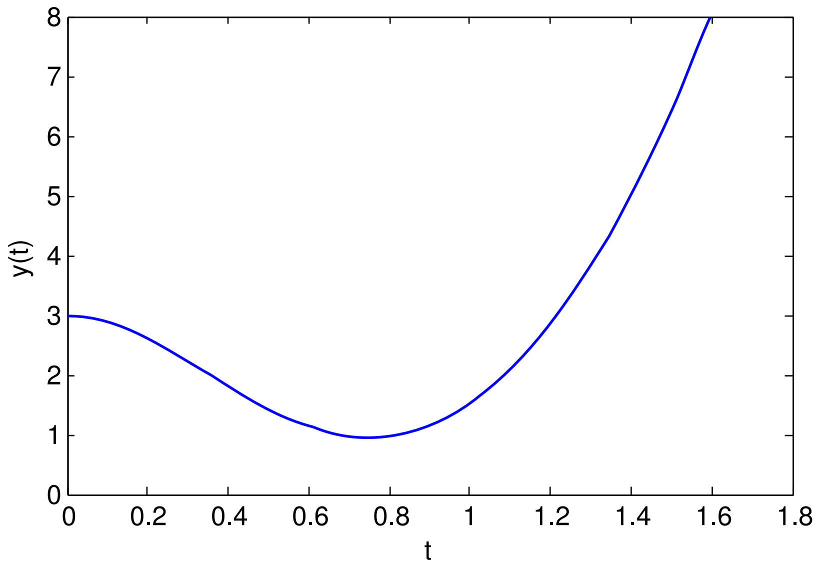

FIGURE 5.1. Solution to $\mathrm{D}^2y + 2\mathrm{D}y + 10y = 6e^{2t}$ shown for the initial conditions $y(0) = 3$ and $y'(0) = 0$.

Example. Find a general solution of

$$\mathrm{L}y = \mathrm{D}^2y - 6\mathrm{D}y + 9y = 4e^{3t}\,.$$

Solution. The characteristic polynomial is

$$p(z) = z^2 - 6z + 9 = (z-3)^2\,.$$

It has the double root 3. Hence, a general solution of the associated homogeneous equation is

$$Y_H(t) = c_1 e^{3t} + c_2 t\, e^{3t}\,.$$

To find a particular solution, first notice that the forcing has characteristic form with characteristic $\mu + i\nu = 3$ and degree $d = 0$. Because the characteristic 3 is a double root of $p(z)$, it has multiplicity $m = 2$.

For $\mu + i\nu = 3$ and $m = 2$ the zero degree formula (5.5b) yields

$$\mathrm{L}\left(\frac{t^2 e^{3t}}{p''(3)}\right) = e^{3t}\,.$$

By multiplying this equation by 4 and using the fact that $p''(z) = 2$ we see that a particular solution of $\mathrm{L}(y) = 4e^{3t}$ is

$$y_P(t) = 4\,\frac{t^2 e^{3t}}{p''(3)} = 4\,\frac{t^2 e^{3t}}{2} = 2t^2 e^{3t}\,.$$

Therefore a general solution is

$$y = c_1 e^{3t} + c_2 t\, e^{3t} + 2t^2 e^{3t}\,.$$

Remark. Had we failed to notice that the characteristic $\mu + i\nu = 3$ has multiplicity $m = 2$ and tried to apply the zero degree formula for the $m = 0$ case, the resulting division by zero should be a warning that $m > 0$!

In the next examples the forcing has complex characteristic form. These examples will require some complex arithmetic. We will make use of the fact that because L has real coefficients if $Y(t)$ and $f(t)$ are complex-valued functions such that then

$$\mathrm{L}(\mathrm{Re}(Y(t))) = \mathrm{Re}(f(t))\,, \qquad \mathrm{L}(\mathrm{Im}(Y(t))) = \mathrm{Im}(f(t))\,.$$

These two real equations are respectively the real and imaginary parts of the single complex equation $\mathrm{L}(Y(t)) = f(t)$, and are thereby equivalent to that equation.

Example. Find a general solution of

$$\mathrm{L}y = \mathrm{D}^2 y + 2\mathrm{D}y + 10y = \cos(2t)\,.$$

Solution. As before, the characteristic polynomial is

$$p(z) = z^2 + 2z + 10 = (z+1)^2 + 3^2\,.$$

Its roots are $-1 \pm i3$. Hence, a general solution of the associated homogeneous equation is

$$Y_H(t) = c_1 e^{-t}\cos(3t) + c_2 e^{-t}\sin(3t)\,.$$

To find a particular solution, first notice that the forcing has characteristic form with characteristic $\mu + i\nu = i2$ and degree $d = 0$. Because the characteristic $i2$ is not a root of $p(z)$, it has multiplicity $m = 0$.

For $\mu + i\nu = i2$ and $m = 0$ the zero degree formula (5.5a) shows that

$$\mathrm{L}\left(\frac{e^{i2t}}{p(i2)}\right) = e^{i2t}\,.$$

Because L has real coefficients the real part of this equation is

$$\mathrm{L}\left(\mathrm{Re}\left(\frac{e^{i2t}}{p(i2)}\right)\right) = \mathrm{Re}\big(e^{i2t}\big) = \cos(2t)\,.$$

We thereby see that a particular solution of $\mathrm{L}(y) = \cos(2t)$ is

$$\begin{aligned} Y_P(t) &= \mathrm{Re}\left(\frac{e^{i2t}}{p(i2)}\right) = \mathrm{Re}\left(\frac{e^{i2t}}{6+i4}\right) = \mathrm{Re}\left(\frac{6-i4}{6^2+4^2}\,e^{i2t}\right) \\ &= \tfrac{1}{52}\mathrm{Re}\big((6-i4)e^{i2t}\big) = \tfrac{1}{52}\mathrm{Re}\big((6-i4)(\cos(2t)+i\sin(2t))\big) \\ &= \tfrac{6}{52}\cos(2t) + \tfrac{4}{52}\sin(2t)\,. \end{aligned}$$

Therefore a general solution is

$$y = c_1 e^{-t}\cos(3t) + c_2 e^{-t}\sin(3t) + \tfrac{3}{26}\cos(2t) + \tfrac{1}{13}\sin(2t)\,.$$

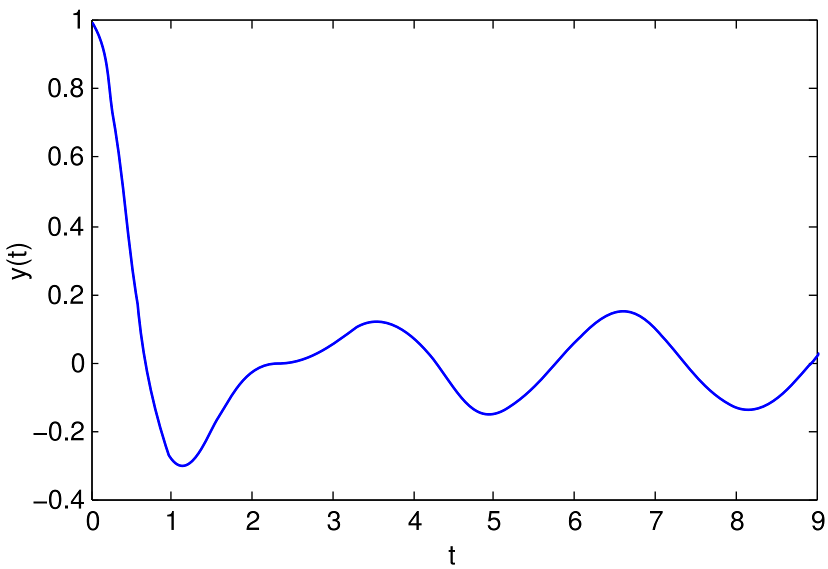

FIGURE 5.2. Solution to $\mathrm{D}^2y + 2\mathrm{D}y + 10y = \cos(2t)$ shown for the initial conditions $y(0) = 1$ and $y'(0) = 0$.

Example. Find a general solution of

$$\mathrm{L}y = \mathrm{D}^2 y + 9y = 4\cos(3t)\,.$$

Solution. This problem has constant coefficients. Its characteristic polynomial is

$$p(z) = z^2 + 9 = z^2 + 3^2\,.$$

Its roots are $\pm i3$. Hence, a general solution of the associated homogeneous equation is

$$Y_H(t) = c_1\cos(3t) + c_2\sin(3t)\,.$$

To find a particular solution, first notice that the forcing has the characteristic form (5.3) with characteristic $\mu + i\nu = i3$ and degree $d = 0$. Because the characteristic $i3$ is a simple root of $p(z)$, it has multiplicity $m = 1$.

For $\mu + i\nu = i3$ and $m = 1$ the zero degree formula (5.5b) shows that

$$\mathrm{L}\left(\frac{t\,e^{i3t}}{p'(i3)}\right) = e^{i3t}\,.$$

Because L has real coefficients the real part of this equation is

$$\mathrm{L}\left(\mathrm{Re}\left(\frac{t\,e^{i3t}}{p'(i3)}\right)\right) = \mathrm{Re}\big(e^{i3t}\big) = \cos(3t)\,.$$

Upon multiplying this by 4 and using the fact that $p'(z) = 2z$, we see that a particular solution of $\mathrm{L}(y) = 4\cos(3t)$ is

$$\begin{aligned} Y_P(t) &= \mathrm{Re}\left(4\frac{t\,e^{i3t}}{p'(i3)}\right) = \mathrm{Re}\left(4\frac{t\,e^{i3t}}{i6}\right) = \frac{4t}{6}\,\mathrm{Re}\big(-i\,e^{i3t}\big) \\ &= \frac{2t}{3}\mathrm{Re}\big(-i(\cos(3t) + i\sin(3t))\big) = \tfrac{2}{3}t\,\sin(3t)\,. \end{aligned}$$

Therefore a general solution is

$$y = c_1\cos(3t) + c_2\sin(3t) + \tfrac{2}{3}t\,\sin(3t)\,.$$

Example. Find a general solution of

$$\mathrm{L}y = \mathrm{D}^2y + 2\mathrm{D}y + 10y = 5e^{-t}\sin(3t)\,.$$

Solution. As before, the characteristic polynomial is

$$p(z) = z^2 + 2z + 10 = (z+1)^2 + 3^2\,.$$

Its roots are $-1 \pm i3$. Hence, a general solution of the associated homogeneous equation is

$$Y_H(t) = c_1e^{-t}\cos(3t) + c_2e^{-t}\sin(3t)\,.$$

To find a particular solution, first notice that the forcing is of the characteristic form (5.3) with characteristic $\mu + i\nu = -1 + i3$ and degree $d = 0$. Because the characteristic $-1 + i3$ is a simple root of $p(z)$, it has multiplicity $m = 1$.

For $\mu + i\nu = -1 + i3$ and $m = 1$ the zero degree formula (5.5b) shows that

$$\mathrm{L}\left(\frac{t\,e^{-t+i3t}}{p'(-1+i3)}\right) = e^{-t+i3t}\,.$$

Because L has real coefficients the imaginary part of this equation is

$$\mathrm{L}\left(\mathrm{Im}\left(\frac{t\,e^{-t+i3t}}{p'(-1+i3)}\right)\right) = \mathrm{Im}\big(e^{-t+i3t}\big) = e^{-t}\sin(3t)\,.$$

Upon multiplying this by 5 and using the fact that $p'(z) = 2(z+1)$, we see that a particular solution of $\mathrm{L}(y) = 5e^{-t}\sin(3t)$ is

$$\begin{aligned} Y_P(t) &= \mathrm{Im}\left(\frac{5t\,e^{-t+i3t}}{p'(-1+i3)}\right) = \mathrm{Im}\left(\frac{5t\,e^{-t}e^{i3t}}{i6}\right) = \frac{5t}{6}\,e^{-t}\mathrm{Im}\big(-i\,e^{i3t}\big) \\ &= \tfrac{5}{6}t\,e^{-t}\mathrm{Im}\big(-i(\cos(3t) + i\sin(3t))\big) = -\tfrac{5}{6}t\,e^{-t}\cos(3t)\,. \end{aligned}$$

Therefore a general solution is

$$y = c_1e^{-t}\cos(3t) + c_2e^{-t}\sin(3t) - \tfrac{5}{6}t\,e^{-t}\cos(3t)\,.$$

5.1.3. *Positive Degree Examples.* When $d > 0$ the method of Key Identity evaluations reduces to $d + 1$ equations that have to be combined into a single equation whose right-hand side is $f(t)$. For example, if the degree $d = 1$ and the characteristic $\mu + i\nu$ has multiplicity $m = 0$ then the two equations are just the Key Identity and its first derivative evaluated at $\mu + i\nu$, which are

$$\begin{aligned} \mathrm{L}\big(e^{(\mu+i\nu)t}\big) &= p(\mu + i\nu)e^{(\mu+i\nu)t}\,, \\ \mathrm{L}\big(t\, e^{(\mu+i\nu)t}\big) &= p(\mu + i\nu)t\, e^{(\mu+i\nu)t} + p'(\mu + i\nu)e^{(\mu+i\nu)t}\,, \end{aligned}$$

Because $m = 0$ we know that $p(\mu + i\nu) \neq 0$, the first of these equations recovers the zero degree formula

$$\mathrm{L}\left(\frac{e^{(\mu+i\nu)t}}{p(\mu + i\nu)}\right) = e^{(\mu+i\nu)t}\,,$$

while the second yields

$$\mathrm{L}\left(\frac{t\, e^{(\mu+i\nu)t}}{p(\mu + i\nu)}\right) = t\, e^{(\mu+i\nu)t} + \frac{p'(\mu + i\nu)}{p(\mu + i\nu)}\, e^{(\mu+i\nu)t}\,.$$

Upon multiplying the zero degree formula by $p'(\mu + i\nu)/p(\mu + i\nu)$ and subtracting it from one just above we obtain

$$\mathrm{L}\left(\frac{t\, e^{(\mu+i\nu)t}}{p(\mu + i\nu)} - \frac{p'(\mu + i\nu)}{p(\mu + i\nu)}\, \frac{e^{(\mu+i\nu)t}}{p(\mu + i\nu)}\right) = t\, e^{(\mu+i\nu)t}\,.$$

By linearly combining the real and imaginary parts of this formula with the real and imaginary parts of the zero degree formula we can obtain an explicit particular solution for any forcing $f(t)$ of characteristic form that has degree $d = 1$ and characteristic $\mu + i\nu$ with multiplicity $m = 0$.

The best approach to using the method of Key Identity evaluations is to mimic the steps that we used to derive the above formulas rather than remembering the formulas themselves. That approach works when $d > 1$ as well as when $m > 0$. We now illustrate this approach.

Example. Find a general solution of

$$\mathrm{L}y = \mathrm{D}^2 y + 2\mathrm{D}y + 10y = 4te^{2t}\,.$$

Solution. As before the characteristic polynomial is

$$p(z) = z^2 + 2z + 10 = (z + 1)^2 + 3^2\,.$$

Its roots are $-1 \pm i3$. Hence, a general solution of the associated homogeneous equation is

$$Y_H(t) = c_1 e^{-t}\cos(3t) + c_2 e^{-t}\sin(3t)\,.$$

To find a particular solution, first notice that the forcing has characteristic form with characteristic $\mu + i\nu = 2$ and degree $d = 1$. Because the characteristic 2 is not a root of $p(z)$, it has multiplicity $m = 0$.

Because $m + d = 1$, we will only need the Key Identity and its first derivative with respect to z:

$$\begin{aligned} \mathrm{L}\big(e^{zt}\big) &= (z^2 + 2z + 10)\, e^{zt} \,, \\ \mathrm{L}\big(t\, e^{zt}\big) &= (z^2 + 2z + 10)\, t\, e^{zt} + (2z + 2)\, e^{zt} \,. \end{aligned}$$

Evaluate these at $z = 2$ to obtain

$$\mathrm{L}\big(e^{2t}\big) = 18e^{2t} \,, \qquad \mathrm{L}\big(t\, e^{2t}\big) = 18\, t\, e^{2t} + 6\, e^{2t} \,.$$

Because we want to isolate the $t\, e^{2t}$ term on the right-hand side, subtract one-third the first equation from the second to get

$$\mathrm{L}\big(t\, e^{2t} - \tfrac{1}{3}e^{2t}\big) = \mathrm{L}\big(t\, e^{2t}\big) - \tfrac{1}{3}\mathrm{L}\big(e^{2t}\big) = 18\, t\, e^{2t} \,.$$

After multiplying this by $\frac{2}{9}$ we can read off that

$$Y_P(t) = \tfrac{2}{9} t e^{2t} - \tfrac{2}{27} e^{2t} \,.$$

Therefore a general solution is

$$y = c_1 e^{-t} \cos(3t) + c_2 e^{-t} \sin(3t) + \tfrac{2}{9} t e^{2t} - \tfrac{2}{27} e^{2t} \,.$$

Example. Find a general solution of

$$\mathrm{L}y = \mathrm{D}^2 y + 4y = t\, \cos(2t) \,.$$

Solution. This problem has constant coefficients. Its characteristic polynomial is

$$p(z) = z^2 + 4 = z^2 + 2^2 \,.$$

Its roots are $\pm i2$. Hence,

$$Y_H(t) = c_1 \cos(2t) + c_2 \sin(2t) \,.$$

To find a particular solution, first notice that the forcing has characteristic form with characteristic $\mu + i\nu = i2$ and degree $d = 1$. Because the characteristic $i2$ is a simple root of $p(z)$, it has multiplicity $m = 1$.

Because $m + d = 2$, we will need the Key Identity and its first two derivatives with respect to z:

$$\begin{aligned} \mathrm{L}\big(e^{zt}\big) &= (z^2 + 4) e^{zt} \,, \\ \mathrm{L}\big(t\, e^{zt}\big) &= (z^2 + 4) t\, e^{zt} + 2z\, e^{zt} \,, \\ \mathrm{L}\big(t^2 e^{zt}\big) &= (z^2 + 4) t^2 e^{zt} + 4z\, t\, e^{zt} + 2e^{zt} \,. \end{aligned}$$

Because $m = 1$, we evaluate the first and second derivative of the Key Identity at $z = i2$ to obtain

$$\mathrm{L}\big(t\, e^{i2t}\big) = i4\, e^{i2t} \,, \qquad \mathrm{L}\big(t^2 e^{i2t}\big) = i8\, t\, e^{i2t} + 2e^{i2t} \,.$$

Because $t\, \cos(2t) = \mathrm{Re}(t\, e^{i2t})$, we want to isolate the $t\, e^{i2t}$ term on the right-hand side. This is done by multiplying the second equation by $i\frac{1}{2}$ and adding it to the third to find

$$\mathrm{L}\big((t^2 + i\tfrac{1}{2}t) e^{i2t}\big) = \mathrm{L}\big(t^2 e^{i2t}\big) + i\tfrac{1}{2}\mathrm{L}\big(t\, e^{i2t}\big) = i8\, t\, e^{i2t} \,.$$

Now divide this by $i8$ to obtain

$$\mathrm{L}\left(\frac{t^2 + i\frac{1}{2}t}{i8}\, e^{i2t}\right) = t\, e^{i2t} \,,$$

from which we read off that

$$Y_P(t) = \mathrm{Re}\left(\frac{t^2 + i\frac{1}{2}t}{i8}\, e^{i2t}\right) = \frac{t}{16}\mathrm{Re}\big((1 - i2t)e^{i2t}\big) = \frac{t}{16}\big(\cos(2t) + 2t\sin(2t)\big)\,.$$

Therefore a general solution is

$$y = c_1\cos(2t) + c_2\sin(2t) + \tfrac{1}{16}t\,\cos(2t) + \tfrac{1}{8}t^2\sin(2t)\,.$$

Remark. The characterisitc $\mu+i\nu = i2$ has multplicity $m = 1$ because $p(z) = z^2+4 = 0$ at $z = i2$. This means the right-hand side of the Key Identity will vanish at $z = i2$, which tells us something we already know, namely, that $\mathrm{L}(e^{i2t}) = 0$. Moreover, it means the term involving $t\,e^{zt}$ on the right-hand side of the derivative of the Key Identity and the term involving t^2e^{zt} on the right-hand side of the second derivative of the Key Identity will also vanish at $z = i2$.

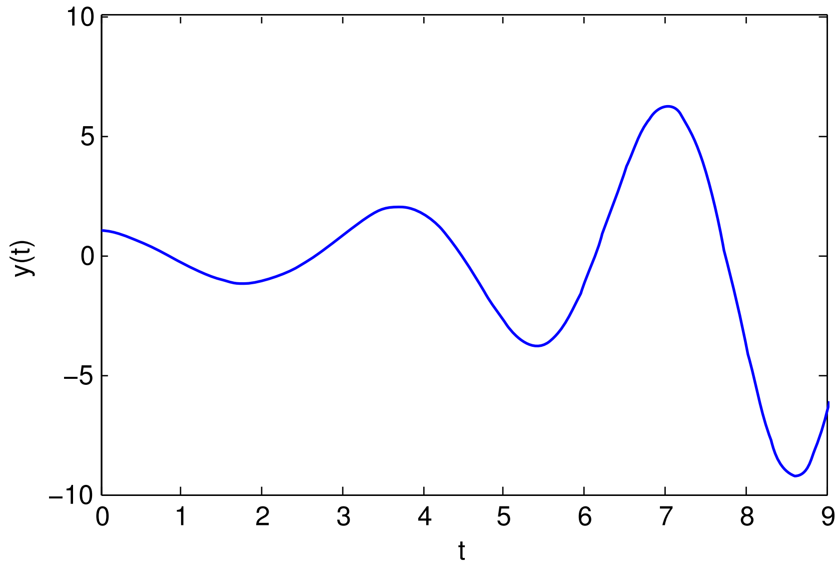

FIGURE 5.3. Solution to $\mathrm{D}^2y + 4y = t\cos(2t)$ shown for the initial conditions $y(0) = 1$ and $y'(0) = 0$

5.1.4. *Why the Method Works.* While the foregoing examples show how the method of Key Identity Evaluations works, they do not show why it works. Specifically, they do not show why it can find a particular solution for every forcing that has characteristic form. We will now show why this is the case. You do not need to know these arguments.

Suppose the forcing $f(t)$ has the characteristic form (5.3) with characteristic $\mu + i\nu$ and degree d. The characteristic form (5.3) can be written as

$$f(t) = \mathrm{Re}\Big(\big(h_0t^d + h_1t^{d-1} + \cdots + h_{d-1}t + h_d\big)\, e^{(\mu+i\nu)t}\Big)\,,$$

where $h_k = f_k - ig_k$ for every $k = 0, 1, \cdots, d$.

Now let $p(z)$ be the characteristic polynomial of the operator L. The k^{th} derivative of the Key Identity with respect to z is

$$\mathrm{L}\big(t^k e^{zt}\big) = \sum_{j=0}^{k} \binom{k}{j} p^{(k-j)}(z) t^j e^{zt}\,,$$

where we recall that the binomial coefficient is defined by

$$\binom{k}{j} = \frac{k!}{(k-j)!\,j!}\,.$$

Suppose that $\mu + i\nu$ is a root of $p(z)$ of multiplicity m. For any C_0, C_1, $\cdots$, C_d we have

$$\begin{aligned}
\mathrm{L}\left(\sum_{k=0}^{d} C_{d-k} t^{m+k} e^{(\mu+i\nu)t}\right) &= \sum_{k=0}^{d} C_{d-k} \mathrm{L}\big(t^{m+k} e^{(\mu+i\nu)t}\big) \\
&= \sum_{k=0}^{d} C_{d-k} \left(\sum_{j=0}^{m+k} \binom{m+k}{j} p^{(m+k-j)}(z) t^j e^{zt}\right) \\
&= \sum_{k=0}^{d} \sum_{j=0}^{k} C_{d-k} \binom{m+k}{j} p^{(m+k-j)}(\mu+i\nu) t^j e^{(\mu+i\nu)t} \\
&= \sum_{j=0}^{d} \left(\sum_{k=j}^{d} \binom{m+k}{j} p^{(m+k-j)}(\mu+i\nu) C_{d-k}\right) t^j e^{(\mu+i\nu)t} \\
&= \sum_{j=0}^{d} h_{d-j} t^j e^{(\mu+i\nu)t}\,,
\end{aligned}$$

where h_0, h_1, $\cdots$, h_d are related to C_0, C_1, $\cdots$, C_d by

$$h_{d-k} = \sum_{k=j}^{d} \binom{m+k}{j} p^{(m+k-j)}(\mu+i\nu) C_{d-k} \qquad \text{for every } k = 0, \cdots, d\,.$$

In particular, these $d+1$ equations have the form

$$\begin{aligned}
h_0 &= \binom{m+d}{d} p^{(m)}(\mu+i\nu) C_0\,, \\
h_1 &= \binom{m+d}{d-1} p^{(m+1)}(\mu+i\nu) C_0 + \binom{m+d-1}{d-1} p^{(m)}(\mu+i\nu) C_1\,, \\
h_2 &= \binom{m+d}{d-2} p^{(m+2)}(\mu+i\nu) C_0 \\
&\quad + \binom{m+d-1}{d-2} p^{(m+1)}(\mu+i\nu) C_1 + \binom{m+d-2}{d-2} p^{(m)}(\mu+i\nu) C_2\,, \\
&\ \ \vdots \\
h_d &= p^{(m+d)}(\mu+i\nu) C_0 + p^{(m+d-1)}(\mu+i\nu) C_1 + \cdots + p^{(m)}(\mu+i\nu) C_d\,,
\end{aligned} \tag{5.6}$$

Because $p^{(m)}(\mu + i\nu) \neq 0$, system (5.6) can be solved for C_0, C_1, $\cdots$, C_d for any given h_0, h_1, $\cdots$, h_d. We first solve the first equation for C_0 and plug the result into the remaining equations. We then solve the second equation for C_1 and plug the result into the remaining equations. This continues until we finally solve the last equation for C_d.

Let C_0, C_1, $\cdots$, C_d be the solution of system (5.6) when we set $h_k = f_k - ig_k$ for every $k = 0, 1, \cdots, d$. Then a particular solution of $\mathrm{L}y = f(t)$ is given by

$$Y_P(t) = \mathrm{Re}\Big(\big(C_0 t^{m+d} + C_1 t^{m+d-1} + \cdots + C_{d-1} t^{m+1} + C_d t^m\big)\, e^{(\mu+i\nu)t}\Big)\,. \tag{5.7}$$

5.2. **Undetermined Coefficients.** This method should only be applied to equation (5.1) when the following two conditions are met.

(1) The differential operator L has constant coefficients.
(2) The forcing $f(t)$ has the characteristic form (5.3) for some characteristic $\mu + i\nu$ and degree d.

These are the same conditions required by the method of Key Identity evalutaions.

5.2.1. *Form for Particular Solutions.* The method of Undetermined Coefficients is based on the observation that if the characteristic $\mu+i\nu$ of the forcing $f(t)$ is not a root of the characteristic polynomial $p(z)$ of the operator L then equation (5.1) has a particular solution of the form

$$\begin{aligned} Y_P(t) = {}& \big(A_0 t^d + A_1 t^{d-1} + \cdots + A_d\big) e^{\mu t}\cos(\nu t) \\ &+ \big(B_0 t^d + B_1 t^{d-1} + \cdots + B_d\big) e^{\mu t}\sin(\nu t)\,, \end{aligned} \tag{5.8}$$

where A_0, A_1, $\cdots$, A_d, and B_0, B_1, $\cdots$, B_d are real constants. Notice that when $\nu = 0$ the terms involving B_0, B_1, $\cdots$, B_d all vanish. More generally, if the characteristic $\mu + i\nu$ is a root of $p(z)$ of multiplicity m then equation (5.1) has a particular solution of the form

$$\begin{aligned} Y_P(t) = {}& \big(A_0 t^{m+d} + A_1 t^{m+d-1} + \cdots + A_d t^m\big) e^{\mu t}\cos(\nu t) \\ &+ \big(B_0 t^{m+d} + B_1 t^{m+d-1} + \cdots + B_d t^m\big) e^{\mu t}\sin(\nu t)\,, \end{aligned} \tag{5.9}$$

where A_0, A_1, $\cdots$, A_d, and B_0, B_1, $\cdots$, B_d are real constants. Notice that when $\nu = 0$ the terms involving B_0, B_1, $\cdots$, B_d all vanish. This case includes the previous one if we once again understand "$\mu + i\nu$ is a root of $p(z)$ of multiplicity 0" to mean that it is not a root of $p(z)$. When we then sets $m = 0$ in (5.9), it reduces to (5.8).

5.2.2. *Determining the Undetermined Coefficients.* Given a nonhomogeneous equation $\mathrm{L}y = f(t)$ in which the forcing $f(t)$ has the characteristic form (5.3) with characteristic $\mu + i\nu$, degree d, and multiplicity m, the method of undetermined coefficients seeks a particular solution $Y_P(t)$ in the form (5.9) with A_0, A_1, $\cdots$, A_d, and B_0, B_1, $\cdots$, B_d as unknowns to be determined. These are the "undetermined coefficients" of the method. There are $2d+2$ unknowns when $\nu \neq 0$, and only $d+1$ unknowns when $\nu = 0$ because in that case the terms involving B_0, B_1, $\cdots$, B_d vanish. These unknowns are determined as follows.

1. Substitute the form (5.9) directly into $\mathrm{L}Y_P$ and collect like terms.

2. Set $\mathrm{L}Y_P = f(t)$ and match the coefficients in front of each of the linearly independent functions that appears on either side. (Examples will make this clearer.)
3. Solve the resulting linear algebraic system for the unknowns in the form (5.9).

This linear algebraic system will consist of either $2d+2$ equations for the $2d+2$ unknowns A_0, A_1, $\cdots$, A_d, and B_0, B_1, $\cdots$, B_d (when $\nu \neq 0$) or $d+1$ equations for the $d+1$ unknowns A_0, A_1, $\cdots$, A_d (when $\nu = 0$). Because these unknowns are the parameters of the family (5.9), this method is also sometimes called "Undetermined Parameters". We do not do so here in order to avoid confusion with the method of "Variation of Parameters" which we will study later.

Remark. The methods of Undetermined Coefficients and Key Identity Evaluations are each fairly painless when m and d are both small and $\nu = 0$. When m and d are both small and $\nu \neq 0$ then Key Identity Evaluations is usually faster. For the problems we will face both m and d will be small, so $m + d$ will seldom be larger than 3, and more commonly be 0, 1, or 2.

5.2.3. *Examples.* In order to contrast the two methods, we will now illustrate the method of Undetermined Coefficients on some of the same examples we had previously treated by Key Identity evaluations.

Example. Find a general solution of

$$\mathrm{L}y = \mathrm{D}^2 y + 2\mathrm{D}y + 10y = 6e^{2t} .$$

Solution. The characteristic polynomial is

$$p(z) = z^2 + 2z + 10 = (z+1)^2 + 9 = (z+1)^2 + 3^2 .$$

Its roots are $-1 \pm i3$. Hence,

$$Y_H(t) = c_1 e^{-t} \cos(3t) + c_2 e^{-t} \sin(3t) .$$

To find a particular solution, first notice that the forcing has the characteristic form (5.3) with characteristic $\mu + i\nu = 2$ and degree $d = 0$. Because the characteristic 2 is not a root of $p(z)$, it has multiplicity $m = 0$.

Because $\mu + i\nu = 2$, $d = 0$, and $m = 0$, we see from (5.9) that Y_P has the form

$$Y_P(t) = Ae^{2t} .$$

Because

$$Y_P'(t) = 2Ae^{2t} , \qquad Y_P''(t) = 4Ae^{2t} ,$$

we see that

$$\begin{aligned} \mathrm{L}Y_P(t) &= Y_P''(t) + 2Y_P'(t) + 10Y_P(t) \\ &= 4Ae^{2t} + 4Ae^{2t} + 10Ae^{2t} = 18Ae^{2t} . \end{aligned}$$

If we set $\mathrm{L}Y_P(t) = 6e^{2t}$ then we see that $18A = 6$, whereby $A = \frac{1}{3}$. Hence,

$$Y_P(t) = \tfrac{1}{3} e^{2t} .$$

Therefore a general solution is

$$y = c_1 e^{-t} \cos(3t) + c_2 e^{-t} \sin(3t) + \tfrac{1}{3} e^{2t} .$$

Example. Find a general solution of

$$\mathrm{L}y = \mathrm{D}^2 y + 2\mathrm{D}y + 10y = 4te^{2t}\,.$$

Solution. As before, the characteristic polynomial is

$$p(z) = z^2 + 2z + 10 = (z+1)^2 + 3^2\,.$$

Its roots are $-1 \pm i3$. Hence,

$$Y_H(t) = c_1 e^{-t}\cos(3t) + c_2 e^{-t}\sin(3t)\,.$$

To find a particular solution, first notice that the forcing has the characteristic form (5.3) with characteristic $\mu + i\nu = 2$ and degree $d = 1$. Because the characteristic 2 is not a root of $p(z)$, it has multiplicity $m = 0$.

Because $\mu + i\nu = 2$, $d = 1$, and $m = 0$, we see from (5.9) that Y_P has the form

$$Y_P(t) = (A_0 t + A_1)e^{2t}\,.$$

Because

$$Y_P'(t) = 2(A_0 t + A_1)e^{2t} + A_0 e^{2t}\,, \qquad Y_P''(t) = 4(A_0 t + A_1)e^{2t} + 4A_0 e^{2t}\,,$$

we see that

$$\begin{aligned}
\mathrm{L}Y_P(t) &= Y_P''(t) + 2Y_P'(t) + 10Y_P(t)\\
&= 4(A_0 t + A_1)e^{2t} + 4A_0 e^{2t} + 4(A_0 t + A_1)e^{2t} + 2A_0 e^{2t} + 10(A_0 t + A_1)e^{2t}\\
&= 18(A_0 t + A_1)e^{2t} + 6A_0 e^{2t}\\
&= 18A_0 te^{2t} + (18A_1 + 6A_0)e^{2t}\,.
\end{aligned}$$

If we set $\mathrm{L}Y_P(t) = 4te^{2t}$ then by equating the coefficients of the linearly independent functions te^{2t} and e^{2t} we see that

$$18A_0 = 4\,, \qquad 18A_1 + 6A_0 = 0\,.$$

Upon solving this linear algebraic system for A_0 and A_1 we first find that $A_0 = \frac{2}{9}$ and then that $A_1 = -\frac{1}{3}A_0 = -\frac{2}{27}$. Hence,

$$Y_P(t) = \tfrac{2}{9}te^{2t} - \tfrac{2}{27}e^{2t}\,.$$

Therefore a general solution is

$$y = c_1 e^{-t}\cos(3t) + c_2 e^{-t}\sin(3t) + \tfrac{2}{9}te^{2t} - \tfrac{2}{27}e^{2t}\,.$$

Example. Find a general solution of

$$\mathrm{L}y = \mathrm{D}^2 y + 2\mathrm{D}y + 10y = \cos(2t)\,.$$

Solution. As before, the characteristic polynomial is

$$p(z) = z^2 + 2z + 10 = (z+1)^2 + 3^2\,.$$

Its roots are $-1 \pm i3$. Hence,

$$Y_H(t) = c_1 e^{-t}\cos(3t) + c_2 e^{-t}\sin(3t)\,.$$

To find a particular solution, first notice that the forcing has the characteristic form (5.3) with characteristic $\mu + i\nu = i2$ and degree $d = 0$. Because the characteristic $i2$ is not a root of $p(z)$, it has mutplicity $m = 0$.

Because $\mu + i\nu = i2$, $d = 0$, and $m = 0$ we see from (5.9) that Y_P has the form

$$Y_P(t) = A\cos(2t) + B\sin(2t)\,.$$

Because

$$Y_P'(t) = -2A\sin(2t) + 2B\cos(2t)\,, \qquad Y_P''(t) = -4A\cos(2t) - 4B\sin(2t)\,,$$

we see that

$$\begin{aligned} \mathrm{L}Y_P(t) &= Y_P''(t) + 2Y_P'(t) + 10Y_P(t) \\ &= -4A\cos(2t) - 4B\sin(2t) - 4A\sin(2t) + 4B\cos(2t) \\ &\quad + 10A\cos(2t) + 10B\sin(2t) \\ &= (6A + 4B)\cos(2t) + (6B - 4A)\sin(2t)\,. \end{aligned}$$

If we set $\mathrm{L}Y_P(t) = \cos(2t)$ then by equating the coefficients of the linearly independent functions $\cos(2t)$ and $\sin(2t)$ we see that

$$6A + 4B = 1\,, \qquad -4A + 6B = 0\,.$$

Upon solving this system we find that $A = \frac{3}{26}$ and $B = \frac{1}{13}$, whereby

$$Y_P(t) = \tfrac{3}{26}\cos(2t) + \tfrac{1}{13}\sin(2t)\,.$$

Therefore a general solution is

$$y = c_1 e^{-t}\cos(3t) + c_2 e^{-t}\sin(3t) + \tfrac{3}{26}\cos(2t) + \tfrac{1}{13}\sin(2t)\,.$$

Example. Find a general solution of

$$\mathrm{L}y = \mathrm{D}^2 y + 4y = t\,\cos(2t)\,.$$

Solution. This problem has constant coefficients. Its characteristic polynomial is

$$p(z) = z^2 + 4 = z^2 + 2^2\,.$$

Its roots are $\pm i2$. Hence,

$$Y_H(t) = c_1\cos(2t) + c_2\sin(2t)\,.$$

To find a particular solution, first notice that the forcing has the characteristic form (5.3) with characteristic $\mu + i\nu = i2$ and degree $d = 1$. Because the characteristic $i2$ is a simple root of $p(z)$, it has multiplicity $m = 1$.

Because $\mu + i\nu = i2$, $d = 1$, and $m = 1$, we see from (5.9) that Y_P has the form

$$Y_P(t) = (A_0 t^2 + A_1 t)\cos(2t) + (B_0 t^2 + B_1 t)\sin(2t)\,.$$

Because

$$\begin{aligned} Y_P'(t) &= -2(A_0t^2 + A_1t)\sin(2t) + (2A_0t + A_1)\cos(2t) \\ &\quad + 2(B_0t^2 + B_1t)\cos(2t) + (2B_0t + B_1)\sin(2t) \\ &= \big(2B_0t^2 + 2(B_1 + A_0)t + A_1\big)\cos(2t) - \big(2A_0t^2 + 2(A_1 - B_0)t - B_1\big)\sin(2t)\,, \\ Y_P''(t) &= -2\big(2B_0t^2 + 2(B_1 + A_0)t + A_1\big)\sin(2t) + \big(4B_0t + 2(B_1 + A_0)\big)\cos(2t) \\ &\quad - 2\big(2A_0t^2 + 2(A_1 - B_0)t - B_1\big)\cos(2t) - \big(4A_0t + 2(A_1 - B_0)\big)\sin(2t) \\ &= -\big(4A_0t^2 + (4A_1 - 8B_0)t - 4B_1 - 2A_0\big)\cos(2t) \\ &\quad - \big(4B_0t^2 + (4B_1 + 8A_0)t + 4A_1 - 2B_0\big)\sin(2t)\,, \end{aligned}$$

we see that

$$\begin{aligned} \mathrm{L}Y_P(t) &= Y_P''(t) + 4Y_P(t) \\ &= -\Big[\big(4A_0t^2 + (4A_1 - 8B_0)t - 4B_1 - 2A_0\big)\cos(2t) \\ &\qquad + \big(4B_0t^2 + (4B_1 + 8A_0)t + 4A_1 - 2B_0\big)\sin(2t)\Big] \\ &\quad + 4\big[(A_0t^2 + A_1t)\cos(2t) + (B_0t^2 + B_1t)\sin(2t)\big] \\ &= \big(8B_0t + 4B_1 + 2A_0\big)\cos(2t) - \big(8A_0t + 4A_1 - 2B_0\big)\sin(2t)\,. \end{aligned}$$

If we set $\mathrm{L}Y_P(t) = t\,\cos(2t)$ then by equating the coefficients of the linearly independent functions $\cos(2t)$, $t\,\cos(2t)$, $\sin(2t)$, and $t\,\sin(2t)$, we see that

$$4B_1 + 2A_0 = 0\,, \qquad 8B_0 = 1\,, \qquad 4A_1 - 2B_0 = 0\,, \qquad 8A_0 = 0\,.$$

The solution of this system is $A_0 = 0$, $B_0 = \frac{1}{8}$, $A_1 = \frac{1}{16}$, and $B_1 = 0$, whereby

$$Y_P(t) = \tfrac{1}{16}t\,\cos(2t) + \tfrac{1}{8}t^2\sin(2t)\,.$$

Therefore a general solution is

$$y = c_1\cos(2t) + c_2\sin(2t) + \tfrac{1}{16}t\,\cos(2t) + \tfrac{1}{8}t^2\sin(2t)\,.$$

Remark. The above example is typical of a case when Key Identity Evaluations is far faster than Undetermined Coefficients. This is because the forcing has a conjugate pair characteristic $\mu \pm i\nu = \pm i2$, positive degree $d = 1$, and small multiplicity $m = 1$. This advantage becomes much more dramatic for larger d. Key Identity Evaluations will usually be as fast or faster than Undetermined Coefficients. If you master both methods you will develop a sense about which one is most efficient for any given problem.

5.2.4. *Why the Method Works.* While the foregoing examples show how the method of Undetermined Coefficients works, they do not show why it works. Specifically, they do not show why there should be a particular solution of the form (5.9) when the forcing has the characteristic form (5.3). We will now show why this is the case. While you do not need to know these arguments, understanding them might help you remember the form (5.9). We will consider two cases.

First, suppose the forcing $f(t)$ has the characteristic form (5.3) with real characteristic μ and degree d — i.e. with $\nu = 0$. Let $p(z)$ be the characteristic polynomial of the operator L. Suppose that μ is a root of $p(z)$ of multiplicity m. Then $p(z)$ can be factored as $p(z) = (z-\mu)^m q(z)$ where $q(\mu) \neq 0$. Observe the characteristic form of $f(t)$ implies that it satisfies the homogeneous linear equation

$$(\mathrm{D} - \mu)^{d+1} f(t) = 0\,.$$

Then every solution of $\mathrm{L}y = f(t)$ also satisfies the homogeneous equation

$$(\mathrm{D} - \mu)^{d+1} \mathrm{L}y = (\mathrm{D} - \mu)^{d+1} f(t) = 0\,.$$

The characteristic polynomial of $(\mathrm{D} - \mu)^{d+1}\mathrm{L}$ is $r(z) = (z-\mu)^{d+1}p(z)$, which factors as

$$r(z) = (z-\mu)^{d+1}p(z) = (z-\mu)^{m+d+1}q(z)\,, \quad \text{where } q(\mu) \neq 0\,.$$

Therefore μ is a root of $r(z)$ of multiplicity $m + d + 1$. All the other roots of $r(z)$ and their multiplicities are determined by the factors of $q(z)$. Therefore a fundamental set of solutions of the homogeneous equation $(\mathrm{D} - \mu)^{d+1}\mathrm{L}y = 0$ is

$$e^{\mu t}\,, \quad t\, e^{\mu t}\,, \quad \cdots\,, \quad t^{m+d} e^{\mu t}\,,$$

plus the solutions generated by the roots of $q(z)$. All of these solutions are also solutions of the homogenous equation $\mathrm{L}w = 0$ except

$$t^m e^{\mu t}\,, \quad t^{m+1} e^{\mu t}\,, \quad \cdots\,, \quad t^{m+d} e^{\mu t}\,.$$

Hence, every solution of $\mathrm{L}y = f(t)$ can be written as $y = Y_H(t) + Y_P(t)$ where $Y_H(t)$ is a solution of the associated homogeneous equation and $Y_P(t)$ has the form (5.5) with $\nu = 0$.

Next, suppose the forcing $f(t)$ has the characteristic form (5.3) with characteristic $\mu + i\nu$ and degree d where $\nu \neq 0$. Let $p(z)$ be the characteristic polynomial of the operator L. Suppose that $\mu + i\nu$ is a root of $p(z)$ of multiplicity m. Then $p(z)$ can be factored as $p(z) = \big((z-\mu)^2 + \nu^2\big)^m q(z)$ where $q(\mu + i\nu) \neq 0$. Observe the characteristic form of $f(t)$ implies that it satisfies the homogeneous linear equation

$$\big((\mathrm{D} - \mu)^2 + \nu^2\big)^{d+1} f(t) = 0\,.$$

Then every solution of $\mathrm{L}y = f(t)$ also satisfies the homogeneous equation

$$\big((\mathrm{D} - \mu)^2 + \nu^2\big)^{d+1} \mathrm{L}y = \big((\mathrm{D} - \mu)^2 + \nu^2\big)^{d+1} f(t) = 0\,.$$

The characteristic polynomial of $\big((\mathrm{D} - \mu)^2 + \nu^2\big)^{d+1}\mathrm{L}$ is $r(z) = \big((z-\mu)^2 + \nu^2\big)^{d+1}p(z)$, which factors as

$$r(z) = \big((z-\mu)^2 + \nu^2\big)^{d+1}p(z) = \big((z-\mu)^2 + \nu^2\big)^{m+d+1}q(z)\,, \quad \text{where } q(\mu + i\nu) \neq 0\,.$$

Therefore $\mu + i\nu$ is a root of $r(z)$ of multiplicity $m + d + 1$. All the other roots of $r(z)$ and their multiplicities are determined by the factors of $q(z)$. Therefore a fundamental set of solutions of the homogeneous equation $\big((\mathrm{D} - \mu)^2 + \nu^2\big)^{d+1}\mathrm{L}y = 0$ is

$$\begin{aligned} &e^{\mu t}\cos(\nu t)\,, \quad t\, e^{\mu t}\cos(\nu t)\,, \quad \cdots\,, \quad t^{m+d} e^{\mu t}\cos(\nu t)\,, \\ &e^{\mu t}\sin(\nu t)\,, \quad t\, e^{\mu t}\sin(\nu t)\,, \quad \cdots\,, \quad t^{m+d} e^{\mu t}\sin(\nu t)\,, \end{aligned}$$

plus the solutions generated by the roots of $q(z)$. All of these solutions are also solutions of the homogenous equation $\mathrm{L}w = 0$ except

$$\begin{aligned} &t^m e^{\mu t}\cos(\nu t)\,, \quad t^{m+1}e^{\mu t}\cos(\nu t)\,, \quad \cdots\,, \quad t^{m+d}e^{\mu t}\cos(\nu t)\,, \\ &t^m e^{\mu t}\sin(\nu t)\,, \quad t^{m+1}e^{\mu t}\sin(\nu t)\,, \quad \cdots\,, \quad t^{m+d}e^{\mu t}\sin(\nu t)\,. \end{aligned}$$

Hence, every solution of $\mathrm{L}y = f(t)$ can be written as $y = Y_H(t) + Y_P(t)$ where $Y_H(t)$ is a solution of the associated homogeneous equation and $Y_P(t)$ has the form (5.9).

Remark. The A_k and B_k in (5.9) are related to the C_k in (5.7) by $C_k = A_k - iB_k$ for every $k = 0, \cdots, d$.

5.3. **Forcings of Composite Characteristic Form.** The methods of Undetermined Coefficients and Key Identity Evaluations can be applied multiple times to construct a particular solution of $\mathrm{L}y = f(t)$ whenever

(1) the differential operator L has constant coefficients,
(2) the forcing $f(t)$ is a sum of terms in the characteristic form (5.3), each with different characteristics.

When the second of these conditions is satisfied the forcing is said to have *composite characteristic form*. The first of these conditions is always easy to verify by inspection. Verification of the second usually can also be done by inspection, but sometimes it might require the use of a trigonometric or some other identity. You should be able to identify when a forcing $f(t)$ can be expressed as a sum of terms that have the characteristic form (5.3), and when it is, to read-off the characteristic and degree of each component.

Example. The forcing of the equation $\mathrm{L}y = \cos(t)^2$ can be written as a sum of terms that have the characteristic form (5.3) by using the identity $\cos(t)^2 = (1 + \cos(2t))/2$. We see that

$$\mathrm{L}y = \cos(t)^2 = \tfrac{1}{2} + \tfrac{1}{2}\cos(2t)\,.$$

Each term on the right-hand side above has the characteristic form (5.3); the first with characteristic $\mu + i\nu = 0$ and degree $d = 0$, and the second with characteristic $\mu + i\nu = i2$ and degree $d = 0$.

Example. The forcing of the equation $\mathrm{L}y = \sin(2t)\cos(3t)$ can be written as a sum of terms that have the characteristic form (5.3) by using the identity

$$\sin(2t)\cos(3t) = \tfrac{1}{2}\big(\sin(3t + 2t) - \sin(3t - 2t)\big) = \tfrac{1}{2}\big(\sin(5t) - \sin(t)\big)\,.$$

We see that

$$\mathrm{L}y = \sin(2t)\cos(3t) = \tfrac{1}{2}\sin(5t) - \tfrac{1}{2}\sin(t)\,.$$

Each term on the right-hand side above has the characteristic form (5.3); the first with characteristic $\mu + i\nu = i5$ and degree $d = 0$, and the second with characteristic $\mu + i\nu = i$ and degree $d = 0$.

Example. The forcing of the equation $\mathrm{L}y = \tan(t)$ cannot be written as a sum of terms in characteristic form (5.3) because every such function is smooth (infinitely differentiable) while $\tan(t)$ is not defined at $t = \frac{\pi}{2} + m\pi$ for every integer m.

Given a nonhomogeneous equation $\mathrm{L}y = f(t)$ in which the forcing $f(t)$ is a sum of terms, each of which has the characteristic form (5.3), we must first identify the characteristic of each term and group all the terms with the same characteristic together. We then decompose $f(t)$ as

$$f(t) = f_1(t) + f_2(t) + \cdots + f_g(t)\,,$$

where each $f_j(t)$ contains all the terms of a given characteristic. Each $f_j(t)$ will then have the characteristic form (5.3) for some characteristic $\mu + i\nu$ and some degree d. Then we can apply either Undetermined Coefficients or Key Identity Evaluations to find particular solutions Y_{jP} to each of

$$\text{(5.10)} \qquad \mathrm{L}Y_{1P}(t) = f_1(t)\,, \quad \mathrm{L}Y_{2P}(t) = f_2(t)\,, \quad \cdots \quad \mathrm{L}Y_{gP}(t) = f_g(t)\,.$$

Then $Y_P(t) = Y_{1P}(t) + Y_{2P}(t) + \cdots + Y_{gP}(t)$ is a particular solution of $\mathrm{L}y = f(t)$.

Example. Find a particular solution of $\mathrm{L}y = \mathrm{D}^4y + 25\mathrm{D}^2y = f(t)$ where

$$f(t) = e^{2t} + 9\cos(5t) + 4t^2e^{2t} - 7t\sin(5t) + 8 - 6t\,.$$

Solution. Decompose $f(t)$ as $f(t) = f_1(t) + f_2(t) + f_3(t)$, where

$$f_1(t) = 8 - 6t\,, \qquad f_2(t) = \big(1 + 4t^2\big)e^{2t}\,, \qquad f_3(t) = 9\cos(5t) - 7t\sin(5t)\,.$$

Here $f_1(t)$, $f_2(t)$, and $f_3(t)$ contain all the terms of $f(t)$ with characteristic 0, 2, and $i5$, respectively. They each have the characteristic form (5.3) with degree 1, 2, and 1 respectively. The characteristic polynomial is $p(z) = z^4 + 25z^2 = z^2(z^2 + 5^2)$, which has roots 0, 0, $-i5$, $i5$. We thereby see that the characteristics 0, 2, and $i5$ have multiplicities 2, 0, and 1 respectively.

The method of Undetermined Coefficients seeks particular solutions of the problems in (5.10) that by (5.9) have the forms

$$\begin{aligned}
Y_{1P}(t) &= A_0t^3 + A_1t^2\,,\\
Y_{2P}(t) &= \big(A_0t^2 + A_1t + A_2\big)e^{2t}\,,\\
Y_{3P}(t) &= \big(A_0t^2 + A_1t\big)\cos(5t) + \big(B_0t^2 + B_1t\big)\sin(5t)\,.
\end{aligned}$$

The method leads to three systems of linear algebraic equations to solve — systems of two equations, three equations, and four equations. We will not solve them here.

Key Identity Evaluations is often the fastest way to solve nonhomogeneous equations whose forcings have composite characteristic form because the Key Identity and its derivatives only have to be computed once. In the problem at hand, $m + d$ for the characteristics 0, 2, and $i5$ are 3, 2, and 2. Therefore we need the Key Identity and its first three derivatives with respect to z:

$$\begin{aligned}
\mathrm{L}\big(e^{zt}\big) &= \big(z^4 + 25z^2\big)\,e^{zt}\,,\\
\mathrm{L}\big(t\,e^{zt}\big) &= \big(z^4 + 25z^2\big)\,t\,e^{zt} + \big(4z^3 + 50z\big)\,e^{zt}\,,\\
\mathrm{L}\big(t^2e^{zt}\big) &= \big(z^4 + 25z^2\big)\,t^2e^{zt} + 2\big(4z^3 + 50z\big)\,t\,e^{zt} + \big(12z^2 + 50\big)\,e^{zt}\,,\\
\mathrm{L}\big(t^3e^{zt}\big) &= \big(z^4 + 25z^2\big)\,t^3e^{zt} + 3\big(4z^3 + 50z\big)\,t^2e^{zt} + 3\big(12z^2 + 50\big)\,t\,e^{zt} + 24z\,e^{zt}\,.
\end{aligned}$$

For the characteristic 0 one has $m = 2$ and $m + d = 3$, so we evaluate the second through third derivative of the Key Identity at $z = 0$ to obtain

$$\mathrm{L}\big(t^2\big) = 50\,, \qquad \mathrm{L}\big(t^3\big) = 150t\,.$$

It follows that $\mathrm{L}\big(\frac{4}{25}t^2 - \frac{1}{25}t^3\big) = 8 - 6t$, whereby $Y_{1P}(t) = \frac{4}{25}t^2 - \frac{1}{25}t^3$.

For the characteristic 2 one has $m = 0$ and $m + d = 2$, so we evaluate the zeroth through second derivative of the Key Identity at $z = 2$ to obtain

$$\begin{aligned} \mathrm{L}\big(e^{2t}\big) &= 116\, e^{2t}\,, \\ \mathrm{L}\big(t\, e^{2t}\big) &= 116\, t\, e^{2t} + 132 e^{2t}\,, \\ \mathrm{L}\big(t^2 e^{2t}\big) &= 116\, t^2 e^{2t} + 264\, t\, e^{2t} + 98\, e^{2t}\,. \end{aligned}$$

We eliminate $t\, e^{2t}$ from the right-hand sides by multiplying the second equation by $\frac{264}{116}$ and subtracting it from the third equation, thereby obtaining

$$\mathrm{L}\big(t^2 e^{2t} - \tfrac{264}{116} t\, e^{2t}\big) = 116\, t^2 e^{2t} + \big(98 - \tfrac{264}{116}132\big)\, e^{2t}\,.$$

Dividing this by 29 gives

$$\mathrm{L}\big(\tfrac{1}{29} t^2 e^{2t} - \tfrac{66}{29^2} t\, e^{2t}\big) = 4\, t^2 e^{2t} + \big(\tfrac{98}{29} - \tfrac{66\cdot 132}{29^2}\big)\, e^{2t}\,.$$

We eliminate e^{2t} from the right-hand side above by multiplying the first equation by $\frac{1}{116}\big(\frac{98}{29} - \frac{66\cdot 132}{29^2}\big)$ and subtracting it from the above equation, thereby obtaining

$$\mathrm{L}\Big(\tfrac{1}{29} t^2 e^{2t} - \tfrac{66}{29^2} t\, e^{2t} - \tfrac{1}{116}\big(\tfrac{98}{29} - \tfrac{66\cdot 132}{29^2}\big)\, e^{2t}\Big) = 4\, t^2 e^{2t}\,.$$

Next, by multiplying the first equation by $\frac{1}{116}$ and adding it to the above equation we obtain

$$\mathrm{L}\Big(\tfrac{1}{29} t^2 e^{2t} - \tfrac{66}{29^2} t\, e^{2t} - \tfrac{1}{116}\big(\tfrac{98}{29} - \tfrac{66\cdot 132}{29^2} - 1\big)\, e^{2t}\Big) = \big(1 + 4\, t^2\big)\, e^{2t}\,,$$

whereby $Y_{2P}(t) = \frac{1}{29} t^2 e^{2t} - \frac{66}{29^2} t\, e^{2t} - \frac{1}{116}\big(\frac{98}{29} - \frac{66\cdot 132}{29^2} - 1\big)\, e^{2t}$.

For the characteristic $i5$ we have $m = 1$ and $m + d = 2$, so we evaluate the first through second derivative of the Key Identity at $z = i5$ to obtain

$$\mathrm{L}\big(t\, e^{i5t}\big) = -i250\, e^{i5t}\,, \qquad \mathrm{L}\big(t^2 e^{i5t}\big) = -i2\cdot 250\, t\, e^{i5t} - 250\, e^{i5t}\,.$$

Upon multiplying the first equation by i and adding it to the second we find that

$$\mathrm{L}\big(t^2 e^{i5t} + it\, e^{i5t}\big) = -i2\cdot 250\, t\, e^{i5t}\,.$$

The first equation and the above equation imply

$$\mathrm{L}\big(i\tfrac{1}{250} t\, e^{i5t}\big) = e^{i5t}\,, \qquad \mathrm{L}\big(\tfrac{1}{500} t^2 e^{i5t} + i\tfrac{1}{500} t\, e^{i5t}\big) = -i\, t\, e^{i5t}\,.$$

The real parts of the above equations are

$$\mathrm{L}\big(-\tfrac{1}{250} t\, \sin(5t)\big) = \cos(5t)\,, \qquad \mathrm{L}\big(\tfrac{1}{500} t^2 \cos(5t) - \tfrac{1}{500} t\, \sin(5t)\big) = t\, \sin(5t)\,.$$

This implies that

$$\mathrm{L}\big(-\tfrac{9}{250} t\, \sin(5t) - \tfrac{7}{500} t^2 \cos(5t) + \tfrac{7}{500} t\, \sin(5t)\big) = 9\cos(5t) - 7t\, \sin(5t)\,,$$

whereby $Y_{3P}(t) = -\frac{11}{500} t\, \sin(5t) - \frac{7}{500} t^2 \cos(5t)$.

Finally, putting all the components together, a particular solution is

$$\begin{aligned} Y_P(t) &= Y_{1P}(t) + Y_{2P}(t) + Y_{3P}(t) \\ &= \tfrac{4}{25} t^2 - \tfrac{1}{25} t^3 + \tfrac{1}{29} t^2 e^{2t} - \tfrac{66}{29^2} t\, e^{2t} - \tfrac{1}{116}\big(\tfrac{98}{29} - \tfrac{66\cdot 132}{29^2} - 1\big)\, e^{2t} \\ &\quad - \tfrac{11}{500} t\, \sin(5t) - \tfrac{7}{500} t^2 \cos(5t)\,. \end{aligned}$$

5.4. **Green Functions: Constant Coefficient Case.** This method can be used to construct a particular solution of an n^{th}-order nonhomogeneous linear ODE

$$\mathrm{L}y = f(t) \tag{5.11}$$

whenever the differential operator L has constant coefficients and is in normal form,

$$\mathrm{L} = \mathrm{D}^n + a_1\mathrm{D}^{n-1} + \cdots + a_{n-1}\mathrm{D} + a_n \,. \tag{5.12}$$

Specifically, for any initial time t_I the particular solution $y = Y_P(t)$ of (5.11) that satisfies the initial condition $Y_P(t_I) = Y_P'(t_I) = \cdots = Y_P^{(n-1)}(t_I) = 0$ is given by

$$Y_P(t) = \int_{t_I}^{t} g(t-s)f(s)\,\mathrm{d}s \,, \tag{5.13}$$

where $g(t)$ is the solution of the homogeneous initial-value problem

$$\mathrm{L}g = 0\,, \qquad g(0) = 0\,, \quad g'(0) = 0\,, \quad \cdots \quad g^{(n-2)}(0) = 0\,, \quad g^{(n-1)}(0) = 1\,. \tag{5.14}$$

The function g is called the *Green function* associated with the operator L. Solving the initial-value problem (5.14) for the Green function is not difficult when the roots of the characteristic polynomials can be found. The method thereby reduces the problem of finding a particular solution $Y_P(t)$ for any forcing $f(t)$ to that of evaluating the integral in (5.13). However, evaluating this integral explicitly can be quite difficult or impossible. In such cases the answer might be expressed in terms of a definite integral.

5.4.1. *Examples.* Before we verify that $Y_P(t)$ given by (5.13) is a solution of (5.11), let us illustrate how the method works with a few examples.

Example. Find a general solution of

$$\mathrm{L}y = \mathrm{D}^2 y - y = \frac{2}{e^t + e^{-t}} \,.$$

Solution. The operator L has constant coefficients and is already in normal form. Its characteristic polynomial is given by $p(z) = z^2 - 1 = (z-1)(z+1)$, which has roots ± 1. Therefore a general solution of the associated homogeneous equation is

$$Y_H(t) = c_1 e^t + c_2 e^{-t} \,.$$

By (5.14) the Green function g associated with L is the solution of the initial-value problem

$$\mathrm{D}^2 g - g = 0\,, \qquad g(0) = 0\,, \quad g'(0) = 1\,.$$

Set $g(t) = c_1 e^t + c_2 e^{-t}$. The first initial condition implies $g(0) = c_1 + c_2 = 0$. Because $g'(t) = c_1 e^t - c_2 e^{-t}$, the second condition implies $g'(0) = c_1 - c_2 = 1$. Upon solving these equations we find that $c_1 = \frac{1}{2}$ and $c_2 = -\frac{1}{2}$. Therefore the Green function is $g(t) = \frac{1}{2}(e^t - e^{-t}) = \sinh(t)$.

The particular solution given by (5.13) with $t_I = 0$ is then

$$Y_P(t) = \int_0^t \frac{e^{t-s} - e^{-t+s}}{e^s + e^{-s}}\,\mathrm{d}s = e^t \int_0^t \frac{e^{-s}}{e^s + e^{-s}}\,\mathrm{d}s - e^{-t} \int_0^t \frac{e^s}{e^s + e^{-s}}\,\mathrm{d}s \,.$$

The definite integrals in the above expression can be evaluated as

$$\int_0^t \frac{e^{-s}}{e^s+e^{-s}}\,\mathrm{d}s = \int_0^t \frac{e^{-2s}}{1+e^{-2s}}\,\mathrm{d}s = -\tfrac{1}{2}\log\big(1+e^{-2s}\big)\Big|_0^t = -\tfrac{1}{2}\log\left(\frac{1+e^{-2t}}{2}\right),$$
$$\int_0^t \frac{e^{s}}{e^s+e^{-s}}\,\mathrm{d}s = \int_0^t \frac{e^{2s}}{e^{2s}+1}\,\mathrm{d}s = \tfrac{1}{2}\log\big(e^{2s}+1\big)\Big|_0^t = \tfrac{1}{2}\log\left(\frac{e^{2t}+1}{2}\right).$$

The above expression for $Y_P(t)$ thereby becomes

$$Y_P(t) = -\tfrac{1}{2}e^t\log\left(\frac{1+e^{-2t}}{2}\right) - \tfrac{1}{2}e^{-t}\log\left(\frac{e^{2t}+1}{2}\right).$$

Therefore a general solution is $y = Y_H(t) + Y_P(t)$ where $Y_H(t)$ and $Y_P(t)$ are given above.

Remark. Notice that in the above example the definite integral in the expression for $Y_P(t)$ given by (5.13) splits into two definite integrals over s whose integrands do not involve t. This kind of splitting always happens. In general, if L is an n^{th}-order operator then the expression for $Y_P(t)$ given by (5.13) always splits into n definite integrals over s whose integrands do not involve t. To do this when the Green function involves terms like $e^{\mu t}\cos(\nu t)$ or $e^{\mu t}\sin(\nu t)$ requires the use of the trigonometric identities

$$\begin{aligned}\cos(\phi-\psi) &= \cos(\phi)\cos(\psi)+\sin(\phi)\sin(\psi)\,,\\ \sin(\phi-\psi) &= \sin(\phi)\cos(\psi)-\cos(\phi)\sin(\psi)\,.\end{aligned} \tag{5.15}$$

You should be familiar with these identities.

Example. Find a general solution of

$$\mathrm{L}y = \mathrm{D}^2y + 9y = \frac{27}{16+9\sin(3t)^2}\,.$$

Solution. The operator L has constant coefficients and is already in normal form. Its characteristic polynomial is given by $p(z) = z^2+9 = z^2+3^2$, which has roots $\pm i3$. Therefore a general solution of the associated homogeneous equation is

$$Y_H(t) = c_1\cos(3t) + c_2\sin(3t)\,.$$

By (5.14) the Green function g associated with L is the solution of the initial-value problem

$$\mathrm{D}^2g + 9g = 0\,, \qquad g(0)=0\,, \quad g'(0)=1\,.$$

Set $g(t) = c_1\cos(3t)+c_2\sin(3t)$. The first initial condition implies $g(0)=c_1=0$, whereby $g(t)=c_2\sin(3t)$. Because $g'(t)=3c_2\cos(3t)$, the second condition implies $g'(0)=3c_2=1$, whereby $c_2=\frac{1}{3}$. Therefore the Green function is $g(t)=\frac{1}{3}\sin(3t)$.

The particular solution given by (5.13) with $t_I=0$ is then

$$Y_P(t) = \int_0^t \sin\big(3(t-s)\big)\frac{9}{16+9\sin(3s)^2}\,\mathrm{d}s\,.$$

By (5.15) with $\phi = 3t$ and $\psi = 3s$, we see $\sin\big(3(t-s)\big) = \sin(3t)\cos(3s) - \cos(3t)\sin(3s)$. We can use this to express $Y_P(t)$ as

$$Y_P(t) = \sin(3t)\int_0^t \frac{9\cos(3s)}{16+9\sin(3s)^2}\,\mathrm{d}s - \cos(3t)\int_0^t \frac{9\sin(3s)}{16+9\sin(3s)^2}\,\mathrm{d}s\,.$$

The definite integrals in the above expression can be evaluated as

$$\begin{aligned}
\int_0^t \frac{9\cos(3s)}{16+9\sin(3s)^2}\,\mathrm{d}s &= \int_0^t \frac{\frac{9}{16}\cos(3s)}{1+\frac{9}{16}\sin(3s)^2}\,\mathrm{d}s \\
&= \tfrac{1}{4}\tan^{-1}\big(\tfrac{3}{4}\sin(3s)\big)\Big|_0^t = \tfrac{1}{4}\tan^{-1}\big(\tfrac{3}{4}\sin(3t)\big)\,. \\
\int_0^t \frac{9\sin(3s)}{16+9\sin(3s)^2}\,\mathrm{d}s &= \int_0^t \frac{9\sin(3s)}{25-9\cos(3s)^2}\,\mathrm{d}s = \int_0^t \frac{\frac{9}{25}\sin(3s)}{1-\frac{9}{25}\cos(3s)^2}\,\mathrm{d}s \\
&= -\tfrac{1}{10}\log\left(\frac{1+\frac{3}{5}\cos(3s)}{1-\frac{3}{5}\cos(3s)}\right)\Bigg|_0^t = -\tfrac{1}{10}\log\left(\frac{1+\frac{3}{5}\cos(3t)}{1-\frac{3}{5}\cos(3t)}\frac{\frac{2}{5}}{\frac{8}{5}}\right)\,.
\end{aligned}$$

Here the first integral has the form

$$\tfrac{1}{4}\int \frac{\mathrm{d}u}{1+u^2} = \tfrac{1}{4}\tan^{-1}(u) + C\,, \qquad \text{where } u = \tfrac{3}{4}\sin(3s)\,,$$

while by using partial fractions we see that the second has the form

$$-\tfrac{1}{5}\int \frac{\mathrm{d}u}{1-u^2} = -\tfrac{1}{10}\log\left(\frac{1+u}{1-u}\right) + C\,, \qquad \text{where } u = \tfrac{3}{5}\cos(3s)\,.$$

The above expression for $Y_P(t)$ thereby becomes

$$Y_P(t) = \tfrac{1}{4}\sin(3t)\tan^{-1}\big(\tfrac{3}{4}\sin(3t)\big) + \tfrac{1}{10}\cos(3t)\log\left(\frac{5+3\cos(3t)}{5-3\cos(3t)}\frac{1}{4}\right)\,.$$

Therefore a general solution is $y = Y_H(t) + Y_P(t)$ where $Y_H(t)$ and $Y_P(t)$ are given above.

Remark. One can evaluate any integral whose integrand is a rational function of sine and cosine. The integrals in the above example are of this type. The next example illustrates what happens in most instances when the Green function method is applied — namely, the integrals that arise cannot be evaluated analytically.

Example. Find a general solution of

$$\mathrm{L}y = \mathrm{D}^2y + 2\mathrm{D}y + 5y = \frac{1}{1+t^2}\,.$$

Solution. The operator L has constant coefficients and is already in normal form. Its characteristic polynomial is given by $p(z) = z^2 + 2z + 5 = (z+1)^2 + 2^2$, which has roots $-1 \pm i2$. Therefore a general solution of the associated homogeneous equation is

$$Y_H(t) = c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t)\,.$$

By (5.14) the Green function g associated with L is the solution of the initial-value problem

$$\mathrm{D}^2 g + 2\mathrm{D}g + 5g = 0\,, \qquad g(0) = 0\,, \quad g'(0) = 1\,.$$

Set $g(t) = c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t)$. The first initial condition implies $g(0) = c_1 = 0$, whereby $g(t) = c_2 e^{-t}\sin(2t)$. Because $g'(t) = 2c_2 e^{-t}\cos(2t) - c_2 e^{-t}\sin(2t)$, the second condition implies $g'(0) = 2c_2 = 1$, whereby $c_2 = \frac{1}{2}$. Therefore the Green function is $g(t) = \frac{1}{2}e^{-t}\sin(2t)$.

The particular solution given by (5.13) with $t_I = \pi$ is then

$$Y_P(t) = \int_\pi^t \tfrac{1}{2} e^{-t+s}\sin\big(2(t-s)\big)\frac{1}{1+s^2}\,\mathrm{d}s\,.$$

By (5.15) with $\phi = 2t$ and $\psi = 2s$, we see $\sin\big(2(t-s)\big) = \sin(2t)\cos(2s) - \cos(2t)\sin(2s)$. We can use this to express $Y_P(t)$ as

$$Y_P(t) = \tfrac{1}{2}e^{-t}\sin(2t)\int_\pi^t \frac{e^s\cos(2s)}{1+s^2}\,\mathrm{d}s - \tfrac{1}{2}e^{-t}\cos(2t)\int_\pi^t \frac{e^s\sin(2s)}{1+s^2}\,\mathrm{d}s\,.$$

The above definite integrals cannot be evaluated analytically. Whenever this is the case, the answer can be left in terms of the integrals. Therefore a general solution is $y = Y_H(t) + Y_P(t)$ where $Y_H(t)$ and $Y_P(t)$ are given above.

Remark. The Green function method should never be used whenever the methods of Undetermined Coefficients and Key Identity Evaluations can be applied. For example, for the equation

$$\mathrm{L}y = \mathrm{D}^2 y + 2\mathrm{D}y + 5y = t\,,$$

the Green function method leads to the expression

$$Y_P(t) = \tfrac{1}{2}e^{-t}\sin(2t)\int_0^t e^s\cos(2s)s\,\mathrm{d}s - \tfrac{1}{2}e^{-t}\cos(2t)\int_0^t e^s\sin(2s)s\,\mathrm{d}s\,.$$

The evaluation of these integrals requires several integration-by-parts. The time it would take to do these integrals is much longer than the time it would take to carry out either of the other two methods, both of which quickly yield $Y_p(t) = \frac{1}{5}t - \frac{2}{25}$!

5.4.2. *Why the Method Works.* Now let us verify that $Y_P(t)$ given by (5.13) indeed always gives a solution of (5.11) when $g(t)$ is the solution of the initial-value problem (5.14). We will use the fact from multivariable calculus that for any continuously differentiable $K(t,s)$ we have

$$\mathrm{D}\int_{t_I}^t K(t,s)\,\mathrm{d}s = K(t,t) + \int_{t_I}^t \partial_t K(t,s)\,\mathrm{d}s\,, \qquad \text{where} \quad \mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}t}\,.$$

Because $g(0) = 0$, we see from (5.13) that

$$\mathrm{D}Y_P(t) = g(0)f(t) + \int_{t_I}^t \mathrm{D}g(t-s)f(s)\,\mathrm{d}s = \int_{t_I}^t \mathrm{D}g(t-s)f(s)\,\mathrm{d}s\,.$$

If $2 < n$ then because $\mathrm{D}g(0) = g'(0) = 0$, we see from the above that

$$\mathrm{D}^2 Y_P(t) = g'(0)f(t) + \int_{t_I}^t \mathrm{D}^2 g(t-s)f(s)\,\mathrm{d}s = \int_{t_I}^t \mathrm{D}^2 g(t-s)f(s)\,\mathrm{d}s\,.$$

If we continue to argue this way then because $\mathrm{D}^{k-1}g(0) = g^{(k-1)}(0) = 0$ for $k < n$, we see that for every $k < n$

$$\mathrm{D}^k Y_P(t) = g^{(k-1)}(0) f(t) + \int_{t_I}^{t} \mathrm{D}^k g(t-s) f(s)\, \mathrm{d}s = \int_{t_I}^{t} \mathrm{D}^k g(t-s) f(s)\, \mathrm{d}s\,.$$

Similarly, because $\mathrm{D}^{n-1}g(0) = g^{(n-1)}(0) = 1$, we see that

$$\mathrm{D}^n Y_P(t) = g^{(n-1)}(0) f(t) + \int_{t_I}^{t} \mathrm{D}^n g(t-s) f(s)\, \mathrm{d}s = f(t) + \int_{t_I}^{t} \mathrm{D}^n g(t-s) f(s)\, \mathrm{d}s\,.$$

Because $\mathrm{L}g(t) = 0$, it follows that $\mathrm{L}g(t-s) = 0$. Then by the above formulas for $\mathrm{D}^k Y_P(t)$, we see that

$$\begin{aligned}
\mathrm{L}Y_P(t) = p(\mathrm{D})Y_P(t) &= \mathrm{D}^n Y_P(t) + a_1 \mathrm{D}^{n-1} Y_P(t) + \cdots + a_{n-1}\mathrm{D}Y_P(t) + a_n Y_P(t) \\
&= f(t) + \int_{t_I}^{t} \mathrm{D}^n g(t-s) f(s)\, \mathrm{d}s + \int_{t_I}^{t} a_1 \mathrm{D}^{n-1} g(t-s) f(s)\, \mathrm{d}s \\
&\quad + \cdots + \int_{t_I}^{t} a_{n-1} \mathrm{D} g(t-s) f(s)\, \mathrm{d}s + \int_{t_I}^{t} a_n g(t-s) f(s)\, \mathrm{d}s \\
&= f(t) + \int_{t_I}^{t} p(\mathrm{D}) g(t-s) f(s)\, \mathrm{d}s \\
&= f(t) + \int_{t_I}^{t} \mathrm{L} g(t-s) f(s)\, \mathrm{d}s = f(t)\,.
\end{aligned}$$

Therefore, $Y_P(t)$ given by (5.13) is a solution of (5.11). Moreover, we see from the above calculations that it is the unique solution of (5.11) that satisfies the initial conditions

$$Y_P(t_I) = 0\,, \quad Y_P'(t_I) = 0\,, \quad \cdots \quad Y_P^{(n-1)}(t_I) = 0\,.$$