UCLA Math 135, Winter 2015
Ordinary Differential Equations

6. Laplace Transform Method

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February 4, 2015

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6. Laplace Transform Method

The Laplace transform allows us to transform an initial-value problem for a linear ordinary differential equation *with constant coefficients* into a linear algebraic equation that can easily be solved. The solution of the initial-value problem can be obtained from the solution of the algebraic equation by taking the so-called inverse Laplace transform.

6.1. **Definition of the Transform.**

The Laplace transform of a function $f(t)$ defined over $t \geq 0$ is another function $\mathcal{L}[f](s)$ that is formally defined by

$$\mathcal{L}[f](s) = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t\,. \tag{6.1}$$

You should recall from calculus that the above definite integral is improper because its upper endpoint is ∞. Because improper definite integrals are defined by limits, the correct definition of the Laplace transform is

$$\mathcal{L}[f](s) = \lim_{T\to\infty} \int_0^T e^{-st} f(t)\,\mathrm{d}t\,, \tag{6.2}$$

provided that the definite integrals over $[0, T]$ appearing in the above limit are proper. The Laplace transform $\mathcal{L}[f](s)$ is defined only at those s for which this limit exists.

Example. Use definition (6.2) to compute $\mathcal{L}[e^{at}](s)$ for any real a.

Solution. From (6.2) we see that for any $s \neq a$ we have

$$\begin{aligned}
\mathcal{L}[e^{at}](s) &= \lim_{T\to\infty} \int_0^T e^{-st} e^{at}\,\mathrm{d}t = \lim_{T\to\infty} \int_0^T e^{(a-s)t}\,\mathrm{d}t \\
&= \lim_{T\to\infty} \frac{e^{(a-s)t}}{a-s}\bigg|_{t=0}^{T} = \lim_{T\to\infty} \left[\frac{1}{s-a} - \frac{e^{(a-s)T}}{s-a}\right] = \begin{cases} \dfrac{1}{s-a} & \text{for } s > a\,, \\ \infty & \text{for } s < a\,, \end{cases}
\end{aligned}$$

while for $s = a$ we have

$$\mathcal{L}[e^{at}](s) = \lim_{T\to\infty} \int_0^T e^{-(s-a)t}\,\mathrm{d}t = \lim_{T\to\infty} \int_0^T \mathrm{d}t = \lim_{T\to\infty} T = \infty\,.$$

Therefore $\mathcal{L}[e^{at}](s)$ is defined only for $s > a$ with

$$\mathcal{L}[e^{at}](s) = \frac{1}{s-a} \quad \text{for } s > a\,.$$

Remark. Notice that $\mathcal{L}[e^{at}](s)$ is defined only for $s > a$. For every $s > a$ it is equal to an expression that is defined for every $s \neq a$, however the equality does not extend to $s < a$. A similar remark will apply to subsequent examples.

Example. Use definition (6.2) to compute $\mathcal{L}[t\, e^{at}](s)$ for any real a.

Solution. From (6.2) we see that for any $s \neq a$ we have

$$\begin{aligned}\mathcal{L}[t\,e^{at}](s) &= \lim_{T\to\infty}\int_0^T t\,e^{-st}e^{at}\,\mathrm{d}t = \lim_{T\to\infty}\int_0^T t\,e^{(a-s)t}\,\mathrm{d}t \\ &= \lim_{T\to\infty}\left(\frac{t}{a-s} - \frac{1}{(a-s)^2}\right)e^{(a-s)t}\bigg|_{t=0}^{T} \\ &= \lim_{T\to\infty}\left[\frac{1}{(s-a)^2} - \left(\frac{T}{s-a} + \frac{1}{(s-a)^2}\right)e^{(a-s)T}\right] = \begin{cases}\dfrac{1}{(s-a)^2} & \text{for } s>a\,,\\ \infty & \text{for } s<a\,,\end{cases}\end{aligned}$$

while for $s = a$ we have

$$\mathcal{L}[t\,e^{at}](s) = \lim_{T\to\infty}\int_0^T t\,e^{-(s-a)t}\,\mathrm{d}t = \lim_{T\to\infty}\int_0^T t\,\mathrm{d}t = \lim_{T\to\infty}\tfrac{1}{2}T^2 = \infty\,.$$

Therefore $\mathcal{L}[t\,e^{at}](s)$ is defined only for $s > a$ with

$$\mathcal{L}[t\,e^{at}](s) = \frac{1}{(s-a)^2} \quad \text{for } s>a\,.$$

Example. Use definition (6.2) to compute $\mathcal{L}[e^{ibt}](s)$ for any real b.

Solution. For $b \neq 0$ we see from (6.2) that for any real s we have

$$\begin{aligned}\mathcal{L}[e^{ibt}](s) &= \lim_{T\to\infty}\int_0^T e^{-st}e^{ibt}\,\mathrm{d}t = \lim_{T\to\infty}\int_0^T e^{-(s-ib)t}\,\mathrm{d}t = \lim_{T\to\infty}\left(-\frac{e^{-(s-ib)t}}{s-ib}\right)\bigg|_{t=0}^{T} \\ &= \lim_{T\to\infty}\left[\frac{1}{s-ib} - \frac{e^{-(s-ib)T}}{s-ib}\right] = \begin{cases}\dfrac{1}{s-ib} & \text{for } s>0\,,\\ \text{undefined} & \text{for } s\leq 0\,.\end{cases}\end{aligned}$$

The case $b = 0$ is identical to our first example with $a = 0$. In either case $\mathcal{L}[e^{ibt}](s)$ is defined only for $s > 0$ with

$$\mathcal{L}[e^{ibt}](s) = \frac{1}{s-ib} \quad \text{for } s>0\,.$$

6.2. **Properties of the Transform.** If we always had to return to the definition of the Laplace transform everytime we wanted to apply it, it would not be easy to use. Rather, we will use the definition to compute the Laplace transform for a few basic functions and to establish some general properties that will allow us to build formulas for more complicated functions.

6.2.1. *Linearity.* The most important property of the Laplace transform $\mathcal{L}$ is that it is a linear operator.

Theorem. If $\mathcal{L}[f](s)$ and $\mathcal{L}[g](s)$ exist for some s then so does $\mathcal{L}[f+g](s)$ and $\mathcal{L}[cf](s)$ for every constant c with

$$\mathcal{L}[f+g](s) = \mathcal{L}[f](s) + \mathcal{L}[g](s)\,, \qquad \mathcal{L}[cf](s) = c\mathcal{L}[f](s)\,. \tag{6.3}$$

Proof. This follows directly from definition (6.2) and the facts that definite integrals and limits depend linearly on their arguments. Specifically, we see that

$$\begin{aligned}\mathcal{L}[f+g](s) &= \lim_{T\to\infty}\int_0^T e^{-st}\big(f(t)+g(t)\big)\,\mathrm{d}t \\ &= \lim_{T\to\infty}\int_0^T e^{-st}f(t)\,\mathrm{d}t + \lim_{T\to\infty}\int_0^T e^{-st}g(t)\,\mathrm{d}t = \mathcal{L}[f](s)+\mathcal{L}[g](s)\,, \\ \mathcal{L}[cf](s) &= \lim_{T\to\infty}\int_0^T e^{-st}cf(t)\,\mathrm{d}t = c\lim_{T\to\infty}\int_0^T e^{-st}f(t)\,\mathrm{d}t = c\mathcal{L}[f](s)\,.\end{aligned}$$

□

Example. Compute $\mathcal{L}[\cos(bt)](s)$ and $\mathcal{L}[\sin(bt)](s)$ for any real $b \neq 0$.

Solution. This can be done by using the Euler identity $e^{ibt} = \cos(bt) + i\sin(bt)$ and the linearity (6.3) of $\mathcal{L}$. Then

$$\mathcal{L}[\cos(bt)](s) + i\mathcal{L}[\sin(bt)](s) = \mathcal{L}[e^{ibt}](s) = \frac{1}{s-ib} = \frac{s+ib}{s^2+b^2} \quad \text{for } s>0\,.$$

By equating the real and imaginary parts above, we see that

$$\begin{aligned}\mathcal{L}[\cos(bt)](s) &= \frac{s}{s^2+b^2} \quad \text{for } s>0\,, \\ \mathcal{L}[\sin(bt)](s) &= \frac{b}{s^2+b^2} \quad \text{for } s>0\,.\end{aligned}$$

6.2.2. *Exponentials and Translations.* Another property of the Laplace transform $\mathcal{L}$ is that it turns multiplication by an exponential in t into a translation of s.

Theorem. If $\mathcal{L}[f](s)$ exists for every $s > \alpha$ and if a is any real number then $\mathcal{L}[e^{at}f(t)](s)$ exists for every $s > \alpha + a$ with

$$\mathcal{L}[e^{at}f(t)](s) = \mathcal{L}[f](s-a) \quad \text{for } s>\alpha+a\,. \tag{6.4}$$

Proof. This follows directly from definition (6.2). Specifically, we see that

$$\mathcal{L}[e^{at}f](s) = \lim_{T\to\infty}\int_0^T e^{-st}e^{at}f(t)\,\mathrm{d}t = \lim_{T\to\infty}\int_0^T e^{-(s-a)t}f(t)\,\mathrm{d}t = \mathcal{L}[f](s-a)\,.$$

□

Examples. From our previous examples and the above theorem we see that

$$\begin{aligned}\mathcal{L}[e^{(a+ib)t}](s) &= \frac{1}{s-a-ib} \qquad \text{for } s>a\,, \\ \mathcal{L}[e^{at}\cos(bt)](s) &= \frac{s-a}{(s-a)^2+b^2} \quad \text{for } s>a\,, \\ \mathcal{L}[e^{at}\sin(bt)](s) &= \frac{b}{(s-a)^2+b^2} \quad \text{for } s>a\,.\end{aligned}$$

Similarly, the Laplace transform turns a translation of t into multiplication by an exponential in s. However, because $\mathcal{L}[f](s)$ only depends upon the values of $f(t)$ over $[0,\infty)$, we have to be careful about what is meant by a translation of t! For example, consider $f_c(t) = f(t-c)$, the translation of $f(t)$ by c.

If $c < 0$ then the graph of f_c is just the graph of f shifted to the left. In that case the values of $f(t)$ over $[0,-c)$ will become the values of $f_c(t)$ over $[c,0)$, on which $\mathcal{L}[f_c](s)$ does not depend. In other words, there can be no simple relation between $\mathcal{L}[f_c](s)$ and $\mathcal{L}[f](s)$ when $c < 0$.

If $c > 0$ then the graph of f_c is just the graph of f shifted to the right. In that case the values of $f(t)$ over $[-c,0)$ will become the values of $f_c(t)$ over $[0,c)$, on which $\mathcal{L}[f_c](s)$ depends. Once again there can be no simple relation between $\mathcal{L}[f_c](s)$ and $\mathcal{L}[f](s)$ when $c < 0$.

However, there is an important difference between the two cases we just considered. When we shifted f to the left, values of $f(t)$ upon which $\mathcal{L}[f](s)$ depends moved outside of $[0,\infty)$. When we shifted f to the right, the values of $f(t)$ upon which $\mathcal{L}[f](s)$ depends stayed over $[0,\infty)$ — the problem was that new values of $f(t)$ moved over $[0,\infty)$. This problem is avoided if before we translate f we multiply it by the *unit step* or *Heaviside* function $u(t)$, which is defined by

$$u(t) = \begin{cases} 1 & \text{for } t \geq 0\,, \\ 0 & \text{for } t < 0\,. \end{cases} \tag{6.5}$$

Because the functions uf and f agree over $[0,\infty)$, it is clear that $\mathcal{L}[uf](s) = \mathcal{L}[f](s)$. We now consider the Laplace transform of $u_c(t)f_c(t) = u(t-c)f(t-c)$ for every $c > 0$.

Theorem. If $\mathcal{L}[f](s)$ exists for every $s > \alpha$ and if c is any positive number then $\mathcal{L}[u(t-c)f(t-c)](s)$ exists for every $s > \alpha$ with

$$\mathcal{L}[u(t-c)f(t-c)](s) = e^{-cs}\mathcal{L}[f](s) \quad \text{for } s > \alpha\,. \tag{6.6}$$

Proof. For every $T > c$ we have

$$\begin{aligned} \int_0^T e^{-st}u(t-c)f(t-c)\,\mathrm{d}t &= \int_c^T e^{-st}f(t-c)\,\mathrm{d}t = e^{-cs}\int_c^T e^{-s(t-c)}f(t-c)\,\mathrm{d}t \\ &= e^{-cs}\int_0^{T-c} e^{-st'}f(t')\,\mathrm{d}t'\,. \end{aligned}$$

Therefore

$$\begin{aligned} \mathcal{L}[u(t-c)f(t-c)](s) &= \lim_{T\to\infty}\int_0^T e^{-st}u(t-c)f(t-c)\,\mathrm{d}t \\ &= e^{-cs}\lim_{T\to\infty}\int_0^{T-c} e^{-st}f(t)\,\mathrm{d}t = e^{-cs}\mathcal{L}[f](s) \quad \text{for } s > \alpha\,. \end{aligned}$$

□

6.3. **Existence and Differentiablity of the Transform.** In each of the above examples the definite integrals over $[0, T]$ that appear in the limit (6.2) were proper. Indeed, we were able to evaluate the definite integrals analytically and then determine the limit (6.2) for every sufficiently large real number s. In this section we identify two properties that when possessed by a function $f(t)$ insure that its Laplace transform exists for every sufficiently large real number s. The first property insures that the definite integrals over $[0, T]$ that appear in the limit (6.2) are all proper. The second property insures that the limit (6.2) of these proper definite integrals exists for every s larger than a certain value. We then use these properties to argue that the Laplace transform $F(s)$ of such an $f(t)$ has derivatives in s of all orders. Moreover, we show that the k^{th} derivative of $F(s)$ is related to the Laplace transform of $t^k f(t)$.

6.3.1. *Piecewise Continuity.* We know from calculus that a definite integral over $[0, T]$ is proper whenever its integrand is:

- bounded over $[0, T]$,
- continuous at all but a finite number of points in $[0, T]$.

Such an integrand is said to be *piecewise continuous* over $[0, T]$. Because e^{-st} is a continuous (and therefore bounded) function of t over every $[0, T]$ for each real s, the definite integrals over $[0, T]$ that appear in the limit (6.2) will be proper whenever $f(t)$ is *piecewise continuous* over every $[0, T]$.

Example. The function

$$f(t) = \begin{cases} 0 & \text{for } 0 \leq t < \pi\,, \\ \cos(t) & \text{for } t \geq \pi\,, \end{cases}$$

is piecewise continuous over every $[0, T]$ because it is clearly bounded over $[0, \infty)$ and its only discontinuity is at the point $t = \pi$.

Example. The so-called *sawtooth* function

$$f(t) = t - k \quad \text{for } k \leq t < k + 1 \text{ where } k = 0, 1, 2, 3, \cdots\,,$$

is piecewise continuous over every $[0, T]$ because it is clearly bounded over $[0, \infty)$ and has discontinuities at the points $t = 1, 2, 3, \cdots$, only a finite number of which lie in each $[0, T]$.

6.3.2. *Exponential Order.* Even if $f(t)$ is piecewise continuous over every $[0, T]$, we still have to give a condition under which the limit (6.2) will exist for certain s. Such a condition is provided by the following definition.

Definition. A function $f(t)$ defined over $[0, \infty)$ is said to be of *exponential order* α *as* $t \to \infty$ provided that for every $\sigma > \alpha$ there exist K_σ and T_σ such that

$$|f(t)| \leq K_\sigma e^{\sigma t} \quad \text{for every } t \geq T_\sigma\,. \tag{6.7}$$

This definition need not be memorized. Rather, we are going to use it to build through examples an understanding of what it means. Roughly speaking, it says that a function is of exponential order α as $t \to \infty$ if its absolute value does not grow faster than $e^{\sigma t}$ as $t \to \infty$ for every $\sigma > \alpha$.

Example. The function e^{at} is of exponential order a as $t \to \infty$ because (6.7) holds with $K_\sigma = 1$ and $T_\sigma = 0$ for every $\sigma > a$.

Example. The functions $\cos(bt)$ and $\sin(bt)$ are of exponential order 0 as $t \to \infty$ because (6.7) holds with $K_\sigma = 1$ and $T_\sigma = 0$ for every $\sigma > 0$.

Example. For every $p > 0$ the function t^p is of exponential order 0 as $t \to \infty$. Indeed, for every $\sigma > 0$ the function $e^{-\sigma t} t^p$ takes on its maximum over $[0, \infty)$ at $t = p/\sigma$, whereby

$$e^{-\sigma t} t^p \leq \left(\frac{p}{e\sigma} \right)^p \quad \text{for every } t \geq 0 \,.$$

Therefore (6.7) holds with $K_\sigma = (\frac{p}{e\sigma})^p$ and $T_\sigma = 0$ for every $\sigma > 0$.

Remark. The last example uses the full power of the definition. It shows that power functions of the form t^p for some $p > 0$ are of exponential order 0 as $t \to \infty$ even though $t^p \to \infty$ as $t \to \infty$. This reflects something that you might recall from calculus — namely, the fact that every exponential function of the form $e^{\sigma t}$ for some $\sigma > 0$ grows faster as $t \to \infty$ than every power function. This fact is sometimes called "expoentials beat powers."

Now that we have understood the exponential order as $t \to \infty$ of the functions e^{at}, $\cos(bt)$, $\sin(bt)$, and t^p, let us ask about the expoential order of combinations of these functions. It can be shown that if functions f and g are of exponential orders α and β respectively as $t \to \infty$ then

- the function $f + g$ is of exponential order $\max\{\alpha, \beta\}$ as $t \to \infty$,
- the function fg is of exponential order $\alpha + \beta$ as $t \to \infty$.

We will not prove these properties. They can be recalled by thinking of the case when f and g are both exponential functions, say $f(t) = e^{at}$ and $g(t) = e^{bt}$. They are easily applied.

Example. For every real a the function $e^{at} + e^{-at}$ is of exponential order $|a|$ as $t \to \infty$. This is because the functions e^{at} and e^{-at} are exponential orders a and $-a$ respectively as $t \to \infty$, and because $|a| = \max\{a, -a\}$.

Example. For every $p > 0$ and every real a and b the function $t^p e^{at} \cos(bt)$ is of exponential order a as $t \to \infty$. This is because the functions t^p, e^{at}, and $\cos(bt)$ are of exponential orders 0, a, and 0 respectively as $t \to \infty$.

6.3.3. *Existence and Differentiablity.* The fact you should know about the existence of the Laplace transform for certain s is the following.

Theorem. Let $f(t)$ be

- piecewise continuous over every $[0, T]$,
- of exponential order α as $t \to \infty$.

Then for every positive integer k the function $t^k f(t)$ has these same properties. The function $F(s) = \mathcal{L}[f](s)$ is defined for every $s > \alpha$. Moreover, $F(s)$ has derivatives of all orders over $s > \alpha$ with its k^{th} derivative satisfying

$$\mathcal{L}[t^k f(t)](s) = (-1)^k F^{(k)}(s) \quad \text{for } s > \alpha \,. \tag{6.8}$$

Proof. Formula (6.8) can be derived formally by differentiating the integrands:

$$\begin{aligned} F(s) &= \int_0^\infty e^{-st} f(t)\, \mathrm{d}t\,, \\ F'(s) &= -\int_0^\infty t\, e^{-st} f(t)\, \mathrm{d}t\,, \\ F''(s) &= \int_0^\infty t^2 e^{-st} f(t)\, \mathrm{d}t\,, \\ &\vdots \\ F^{(k)}(s) &= (-1)^k \int_0^\infty t^k e^{-st} f(t)\, \mathrm{d}t\,. \end{aligned}$$

□

Remark. A correct proof would require a justification of taking the derivatives inside the above imporper integrals. We will not go into those details here. However, we will give an easier proof of the fact that $F(s)$ is defined for $s > \alpha$. The proof uses the direct comparison test for the convergence of improper integrals. That test implies that if $g(t)$ and $h(t)$ are piecewise continuous over every $[0, T]$ such that $|g(t)| \leq h(t)$ for every $t \geq 0$ then

$$\int_0^\infty h(t)\, \mathrm{d}t \quad \text{converges} \qquad \Longrightarrow \qquad \int_0^\infty g(t)\, \mathrm{d}t \quad \text{converges}\,.$$

Let $s > \alpha$ and apply this test to $g(t) = e^{-st} f(t)$. Pick σ so that $\alpha < \sigma < s$. Because $f(t)$ is of exponential order α as $t \to \infty$ and $\sigma > \alpha$ there exist K_σ and T_σ such that (6.7) holds. Because $g(t) = e^{-st} f(t)$ is bounded over $[0, T_\sigma]$ there exists B_σ such that $|g(t)| \leq B_\sigma$ over $[0, T_\sigma]$. It thereby follows that

$$|g(t)| = e^{-st} |f(t)| \leq h(t) \equiv \begin{cases} B_\sigma & \text{for } 0 \leq t < T_\sigma \\ K_\sigma e^{(\sigma - s)t} & \text{for } t \geq T_\sigma\,. \end{cases}$$

Because $s > \sigma$ for this $h(t)$ it can be shown that

$$\int_0^\infty h(t)\, \mathrm{d}t = \lim_{T \to \infty} \int_0^T h(t)\, \mathrm{d}t \quad \text{converges}\,.$$

It follows that the limit in (6.2) converges, whereby $F(s) = \mathcal{L}[f](s)$ is defined at s.

Example. Because for every real a and b we have

$$\mathcal{L}[e^{(a+ib)t}](s) = \frac{1}{s - a - ib} \quad \text{for } s > a\,,$$

it follows from the above theorem that for every nonnegative integer k

$$\mathcal{L}[t^k e^{(a+ib)t}](s) = (-1)^k \frac{\mathrm{d}^k}{\mathrm{d}s^k} \frac{1}{s - a - ib} = \frac{k!}{(s - a - ib)^{k+1}} \quad \text{for } s > a\,.$$

This formula implies that for every real a and b and every nonnegative integer k

$$\begin{aligned}
\mathcal{L}[t^k](s) &= \frac{k!}{s^{k+1}} && \text{for } s > 0\,,\\
\mathcal{L}[t^k e^{at}](s) &= \frac{k!}{(s-a)^{k+1}} && \text{for } s > a\,.\\
\mathcal{L}[t^k e^{at}\cos(bt)](s) &= \mathrm{Re}\bigg(\frac{k!}{(s-a-ib)^{k+1}}\bigg) && \text{for } s > a\,,\\
\mathcal{L}[t^k e^{at}\sin(bt)](s) &= \mathrm{Im}\bigg(\frac{k!}{(s-a-ib)^{k+1}}\bigg) && \text{for } s > a\,.
\end{aligned}$$

6.4. **Transform of Derivatives.** In the previous section we saw that the Laplace transform turns a multiplication by t into a derivative with respect to s. The next result shows the Laplace transform turns a derivative with respect to t into a multiplication by s.

Theorem. Let $f(t)$ be continuous over $[0,\infty)$ and be differentiable at all but a finite number of points of every $[0,T]$. If

- $f(t)$ is of exponential order α as $t \to \infty$,
- $f'(t)$ is piecewise continuous over every $[0,T]$,

then $\mathcal{L}[f'](s)$ is defined for every $s > \alpha$ with

$$\mathcal{L}[f'](s) = s\,\mathcal{L}[f](s) - f(0)\,. \tag{6.9}$$

Proof. Let $s > \alpha$. By definition (6.2), an integration by parts, the fact that $f(t)$ is of exponential order α as $t \to \infty$, and the fact that $\mathcal{L}[f](s)$ exists, we see that

$$\begin{aligned}
\mathcal{L}[f'](s) &= \lim_{T\to\infty}\int_0^T e^{-st} f'(t)\,\mathrm{d}t = \lim_{T\to\infty}\bigg[e^{-st}f(t)\Big|_{t=0}^{T} + s\int_0^T e^{-st}f(t)\,\mathrm{d}t\bigg]\\
&= \lim_{T\to\infty} e^{-sT}f(T) - f(0) + s\lim_{T\to\infty}\int_0^T e^{-st}f(t)\,\mathrm{d}t = -f(0) + s\,\mathcal{L}[f](s)\,.
\end{aligned}$$

□

Example. Let $f(t) = \sin(e^{t^2})$. Because $f(t)$ is bounded, it is of exponential order 0 as $t \to \infty$. Because $f(t)$ is continuous over $[0,\infty)$ and of exponential order 0 as $t \to \infty$, its Laplace transform $\mathcal{L}[\sin(e^{t^2})](s)$ is defined for every $s > 0$, even though it cannot be computed explicitly. Because $f(t)$ is also continuously differentiable over $[0,\infty)$ with $f'(t) = 2te^{t^2}\cos(e^{t^2})$, the above theorem yields

$$\mathcal{L}[2te^{t^2}\cos(e^{t^2})](s) = s\,\mathcal{L}[\sin(e^{t^2})](s) - \sin(1) \quad \text{for } s > 0\,.$$

Remark. Until the last example, every functions that we have shown to have a Laplace transform has been of exponential order as $t \to \infty$. However, $f'(t) = 2te^{t^2}\cos(e^{t^2})$ is not of exponential order as $t \to \infty$, yet its Laplace transform is defined for every $s > 0$. This shows that having an exponential order as $t \to \infty$ is not necessary for a function to have a Laplace transform.

If $f(t)$ is sufficiently differentiable then formula (6.9) can be applied repeatedly. For example, if $f(t)$ is twice differentiable then

$$\begin{aligned}\mathcal{L}[f''](s) = s\,\mathcal{L}[f'](s) - f'(0) &= s\big(s\,\mathcal{L}[f](s) - f(0)\big) - f'(0) \\ &= s^2\mathcal{L}[f](s) - s\,f(0) - f'(0)\,.\end{aligned}$$

If $f(t)$ is thrice differentiable then

$$\begin{aligned}\mathcal{L}[f'''](s) = s\,\mathcal{L}[f''](s) - f''(0) &= s\big(s^2\mathcal{L}[f](s) - s\,f(0) - f'(0)\big) - f''(0) \\ &= s^3\mathcal{L}[f](s) - s^2 f(0) - s\,f'(0) - f''(0)\,.\end{aligned}$$

Proceeding in this way we can use induction to prove the following.

Theorem. Let $f(t)$ be $(n-1)$-times continuously differentiable over $[0,\infty)$ and $f^{(n-1)}$ be differentiable at all but a finite number of points of every $[0,T]$. If

- $f(t)$, $f'(t)$, $\cdots$, $f^{(n-1)}(t)$ are of exponential order α as $t\to\infty$,
- $f^{(n)}(t)$ is piecewise continuous over every interval $[0,T]$,

then $\mathcal{L}[f^{(n)}](s)$ is defined for every $s>\alpha$ with

$$\mathcal{L}[f^{(n)}](s) = s^n\mathcal{L}[f](s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - s\,f^{(n-2)}(0) - f^{(n-1)}(0)\,. \tag{6.10}$$

This means that if we know that a function $y(t)$ is n-times differentiable and that it and its first $n-1$ derivatives are of exponential order as $t\to\infty$ then we have

$$\begin{aligned}\mathcal{L}[y](s) &= Y(s)\,,\\ \mathcal{L}[y'](s) &= s\,\mathcal{L}[y](s) - y(0) = s\,Y(s) - y(0)\,,\\ \mathcal{L}[y''](s) &= s\,\mathcal{L}[y'](s) - y'(0) = s^2Y(s) - s\,y(0) - y'(0)\,,\\ \mathcal{L}[y'''](s) &= s\,\mathcal{L}[y''](s) - y''(0) = s^3Y(s) - s^2y(0) - s\,y'(0) - y''(0)\,,\\ &\;\;\vdots\\ \mathcal{L}[y^{(n)}](s) &= s^nY(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \cdots - s\,y^{(n-2)}(0) - y^{(n-1)}(0)\,.\end{aligned} \tag{6.11}$$

6.5. **Application to Initial-Value Problems.** Because the Laplace transform turns derivatives with respect to t into multiplications by s, it transforms initial-value problems into algebraic problems. Consider the initial-value problem

$$y^{(n)} + a_1y^{(n-1)} + \cdots + a_{n-1}y' + a_ny = f(t)\,, \tag{6.12a}$$

$$y(0) = y_0\,, \qquad y'(0) = y_1\,, \qquad \cdots \qquad y^{(n-1)}(0) = y_{n-1}\,. \tag{6.12b}$$

We will use the following theorem, which we state without proof.

Theorem. Let $f(t)$ be piecewise continuous over $[0,\infty)$ and of exponential order as $t\to\infty$. Then there exists a unique $y(t)$ that is $(n-1)$-times continously differentiable over $[0,\infty)$ such that

- $y^{(n-1)}$ is continuously differentiable at all points in $[0,\infty)$ where f is continuous,
- equation (6.12a) is satisfied at all points in $[0,\infty)$ where f is continuous,
- the initial conditions (6.12b) are satisfied,
- y and its first n derivatives are of exponential order as $t\to\infty$.

Remark. This theorem introduces a new notion of solution for the initial-value problem (6.12). Specifically, it requires that the differential equation (6.12a) be satisfied only at those points in $[0,\infty)$ where f is continuous, rather than at all points in $[0,\infty)$.

This theorem allows us to use the Laplace transform to find $Y(s) = \mathcal{L}[y](s)$ in terms of the inital data y_0, y_1, $\cdots$, y_{n-1}, and the Laplace transform of the forcing, $F(s) = \mathcal{L}[f](s)$. Later we will see how to determine $y(t)$ from $Y(s)$, but here we will illustrate how to compute $Y(s)$.

First, we use the linearity of $\mathcal{L}$ to express the Laplace transform of the differential equation (6.12a) as

$$\mathcal{L}[y^{(n)}] + a_1\mathcal{L}[y^{(n-1)}] + \cdots + a_{n-1}\mathcal{L}[y'] + a_n\mathcal{L}[y] = \mathcal{L}[f]\,.$$

Second, we use (6.11) and the initial conditions (6.12b) to write

$$\begin{aligned}
\mathcal{L}[y](s) &= Y(s)\,,\\
\mathcal{L}[y'](s) &= s\,Y(s) - y_0\,,\\
\mathcal{L}[y''](s) &= s^2Y(s) - s\,y_0 - y_1\,,\\
&\;\;\vdots\\
\mathcal{L}[y^{(n)}](s) &= s^nY(s) - s^{n-1}y_0 - s^{n-2}y_1 - \cdots - s\,y_{n-1} - y_{n-1}\,.
\end{aligned}$$

Third, we compute $F(s) = \mathcal{L}[f](s)$. Fourth, by placing the results of the second and third steps into the Laplace transform of the differential equation obtained in the first step, we see that $Y(s)$ satisfies the linear algebraic equation

$$p(s)Y(s) = q(s) + F(s)\,,$$

where $p(s)$ is the characteristic polynomial

$$p(s) = s^n + a_1s^{n-1} + \cdots + a_{n-1}s + a_n\,.$$

and $q(s)$ is the polynomial given in terms of the initial data by

$$\begin{aligned}
q(s) = {}&\big(s^{n-1} + a_1s^{n-2} + \cdots + a_{n-2}s + a_{n-1}\big)y_0\\
&+ \big(s^{n-2} + a_1s^{n-3} + \cdots + a_{n-3}s + a_{n-2}\big)y_1\\
&+ \cdots + \big(s^2 + a_1s + a_2\big)y_{n-3} + \big(s + a_1\big)y_{n-2} + y_{n-1}\,.
\end{aligned}$$

Finally, we solve the linear algebraic equation for $Y(s)$ to obtain

$$Y(s) = \frac{q(s) + F(s)}{p(s)}\,. \tag{6.13}$$

The hardest of the above steps is the third — namely, computing $F(s) = \mathcal{L}[f](s)$. Often $f(t)$ is a combination of the basic forms whose Laplace transform we have already computed. These basic forms include

$$
\begin{aligned}
\mathcal{L}[t^n](s) &= \frac{n!}{s^{n+1}} && \text{for } s > 0\,, \\
\mathcal{L}[\cos(bt)](s) &= \frac{s}{s^2+b^2} && \text{for } s > 0\,, \\
\mathcal{L}[\sin(bt)](s) &= \frac{b}{s^2+b^2} && \text{for } s > 0\,, \\
\mathcal{L}[e^{at}j(t)](s) &= J(s-a) && \text{where } J(s) = \mathcal{L}[j(t)](s)\,, \\
\mathcal{L}[t^n j(t)](s) &= (-1)^n J^{(n)}(s) && \text{where } J(s) = \mathcal{L}[j(t)](s)\,, \\
\mathcal{L}[u(t-c)j(t-c)](s) &= e^{-cs}J(s) && \text{where } J(s) = \mathcal{L}[j(t)](s)\,, \\
\mathcal{L}[e^{at}t^n](s) &= \frac{n!}{(s-a)^{n+1}} && \text{for } s > a\,, \\
\mathcal{L}[e^{at}\cos(bt)](s) &= \frac{s-a}{(s-a)^2+b^2} && \text{for } s > a\,, \\
\mathcal{L}[e^{at}\sin(bt)](s) &= \frac{b}{(s-a)^2+b^2} && \text{for } s > a\,.
\end{aligned}
\tag{6.14}
$$

These can be used to build a longer table like those found in many textbooks. However, this table is all we need. In fact, its first three entries are the last three for $a = 0$. Alternatively, its last three entries follow from the first three and the fourth. On exams you will be given a similar table, so you do not have to memorize this one. However, you should learn how to use it efficiently.

Example. Find the Laplace transform $Y(s)$ of the solution $y(t)$ of the initial-value problem

$$y' - 2y = e^{5t}\,, \qquad y(0) = 3\,.$$

Solution. By setting $a = 5$ and $n = 0$ in the seventh entry of table (6.14) we see that $\mathcal{L}[e^{5t}](s) = 1/(s-5)$. Therefore the Laplace transform of the initial-value problem is

$$\mathcal{L}[y'](s) - 2\mathcal{L}[y](s) = \mathcal{L}[e^{5t}](s) = \frac{1}{s-5}\,,$$

where we see from (6.11) that

$$\mathcal{L}[y](s) = Y(s)\,, \qquad \mathcal{L}[y'](s) = s\,\mathcal{L}[y](s) - y(0) = s\,Y(s) - 3\,.$$

It follows that

$$(s-2)Y(s) - 3 = \frac{1}{s-5}\,, \qquad \Longrightarrow \qquad Y(s) = \frac{1}{(s-2)(s-5)} + \frac{3}{s-2}\,.$$

Example. Find the Laplace transform $Y(s)$ of the solution $y(t)$ of the initial-value problem

$$y'' - 2y' - 8y = 0\,, \qquad y(0) = 3\,, \quad y'(0) = 7\,.$$

Solution. Here there is no forcing. The Laplace transform of the initial-value problem is

$$\mathcal{L}[y''](s) - 2\mathcal{L}[y'](s) - 8\mathcal{L}[y](s) = 0\,,$$

where we see from (6.11) that

$$\begin{aligned}\mathcal{L}[y](s) &= Y(s)\,,\\ \mathcal{L}[y'](s) &= s\,\mathcal{L}[y](s) - y(0) = s\,Y(s) - 3\,,\\ \mathcal{L}[y''](s) &= s\,\mathcal{L}[y'](s) - y'(0) = s^2 Y(s) - 3s - 7\,.\end{aligned}$$

It follows that

$$(s^2 - 2s - 8)Y(s) - 3s - 1 = 0\,, \qquad \Longrightarrow \qquad Y(s) = \frac{3s+1}{s^2 - 2s - 8}\,.$$

Example. Find the Laplace transform $Y(s)$ of the solution $y(t)$ of the initial-value problem

$$y'' + 4y = \sin(3t)\,, \qquad y(0) = y'(0) = 0\,.$$

Solution. By setting $b = 3$ in the third entry of table (6.14) we see that $\mathcal{L}[\sin(3t)](s) = 3/(s^2+9)$. Therefore the Laplace transform of the initial-value problem is

$$\mathcal{L}[y''](s) + 4\mathcal{L}[y](s) = \mathcal{L}[\sin(3t)](s) = \frac{3}{s^2 + 3^2} = \frac{3}{s^2+9}\,,$$

where we see from (6.11) that

$$\mathcal{L}[y](s) = Y(s)\,, \qquad \mathcal{L}[y''](s) = s^2 Y(s) - sy(0) - y'(0) = s^2 Y(s)\,.$$

It follows that

$$(s^2+4)Y(s) = \frac{3}{s^2+9}\,, \qquad \Longrightarrow \qquad Y(s) = \frac{3}{(s^2+4)(s^2+9)}\,.$$

6.6. **Piecewise-Defined Forcing.** In the previous section we stated that the Laplace transform method can be used to solve initial-value problems of the form

$$\begin{aligned} &y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}y' + a_n y = f(t)\,,\\ &y(0) = y_0\,, \qquad y'(0) = y_1\,, \qquad \cdots \qquad y^{(n-1)}(0) = y_{n-1}\,,\end{aligned} \tag{6.15a}$$

where the forcing $f(t)$ is piecewise-defined over $[0,\infty)$ by a list in the form

$$f(t) = \begin{cases} f_0(t) & \text{for } 0 \le t < c_1\,,\\ f_1(t) & \text{for } c_1 \le t < c_2\,,\\ \quad\vdots & \qquad\quad\vdots\\ f_{m-1}(t) & \text{for } c_{m-1} \le t < c_m\,,\\ f_m(t) & \text{for } c_m \le t < \infty\,,\end{cases} \tag{6.15b}$$

where $0 = c_0 < c_1 < \cdots < c_m < \infty$. We assume that for each $k = 0, 1, \cdots, m-1$ the function f_k is continuous and bounded over $[c_k, c_{k+1})$, and that the function f_m is continuous over $[c_m, \infty)$ and is of exponential order as $t \to \infty$. In this section we show how to compute the Laplace transform $F(s) = \mathcal{L}[f](s)$ for such a function. There are three steps.

The first step is to express $f(t)$ in terms of translations of the unit step $u(t)$. How this is done should become clear once you see that for every $0 \leq c < d$ we have

$$u(t-c) - u(t-d) = \begin{cases} 1 & \text{for } c \leq t < d\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

In other words, the function $u(t-c) - u(t-d)$ is a switch that turns on at $t = c$ and turns off at $t = d$. So for any given function $g(t)$ we have

$$\big(u(t-c) - u(t-d)\big)\, g(t) = \begin{cases} g(t) & \text{for } c \leq t < d\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

This observation allows us to express $f(t)$ as

$$\begin{aligned} f(t) = \big(u(t) - u(t-c_1)\big)\, f_0(t) + \big(u(t-c_1) - u(t-c_2)\big)\, f_1(t) \\ + \cdots + \big(u(t-c_{m-1}) - u(t-c_m)\big)\, f_{m-1}(t) + u(t-c_m)\, f_m(t)\,. \end{aligned}$$

By grouping terms above that involve the same $u(t-c_k)$, we bring $f(t)$ into the form

(6.16) $$f(t) = f_0(t) + u(t-c_1)\, h_1(t) + \cdots + u(t-c_m)\, h_m(t)\,,$$

where $h_k(t) = f_k(t) - f_{k-1}(t)$ for $k = 1, 2, \cdots, m$. This is the form we want. It can be obtained either by carrying out the grouping indicated above or by recalling that each term $u(t-c_k)\, h_k(t)$ appearing in (6.16) simply changes the forcing from $f_{k-1}(t)$ to $f_k(t)$ at time $t = c_k$ because $h_k(t) = f_k(t) - f_{k-1}(t)$.

The idea of the second step is to bring (6.16) into a form that allows us to use the sixth entry in table (6.14). That entry states that $\mathcal{L}[u(t-c)j(t-c)](s) = e^{-cs}\mathcal{L}[j](s)$. Therefore we must recast (6.16) into the form

(6.17a) $$f(t) = f_0(t) + u(t-c_1)\, j_1(t-c_1) + \cdots + u(t-c_m)\, j_m(t-c_m)\,.$$

Each function $j_k(t)$ is obtained from the $h_k(t)$ appearing in (**??**) by

(6.17b) $$j_k(t) = h_k(t+c_k) \quad \text{for } k = 1, 2, \cdots, m\,.$$

Indeed, this formula implies that $j_k(t-c_k) = h_k(t)$, which is why (6.16) becomes (6.17a).

Once we have found all the $j_k(t)$ then the final step is to compute $\mathcal{L}[f_0](s)$ and each $\mathcal{L}[j_k](s)$, and use the fact that the sixth entry of table (6.14) implies

$$\mathcal{L}[u(t-c_k)j_k(t-c_k)](s) = e^{-c_k s}\mathcal{L}[j_k](s) \quad \text{for } k = 1, 2, \cdots, m\,,$$

to compute $\mathcal{L}[f](s)$ as

(6.18) $$\mathcal{L}[f](s) = \mathcal{L}[f_0](s) + e^{-c_1 s}\mathcal{L}[j_1](s) + \cdots + e^{-c_m s}\mathcal{L}[j_m](s)\,.$$

Often we will have to use identities to express $f_0(t)$ and each $j_k(t)$ in forms that allows us to compute their Laplace transforms from table (6.14).

Example. Find the Laplace transform $Y(s)$ of the solution $y(t)$ of the initial-value problem

$$y'' + 4y = f(t)\,, \qquad y(0) = 7\,, \quad y'(0) = 5\,,$$

where

$$f(t) = \begin{cases} t^2 & \text{for } 0 \le t < 2\,, \\ 2t & \text{for } 2 \le t < 4\,, \\ 4 & \text{for } 4 \le t\,. \end{cases}$$

Solution. The Laplace transform of the initial-value problem is

$$\mathcal{L}[y''](s) + 4\mathcal{L}[y](s) = F(s)\,,$$

where $F(s) = \mathcal{L}[f](s)$ and

$$\begin{aligned} \mathcal{L}[y](s) &= Y(s)\,, \\ \mathcal{L}[y'](s) &= s\,\mathcal{L}[y](s) - y(0) = sY(s) - 7\,, \\ \mathcal{L}[y''](s) &= s\,\mathcal{L}[y'](s) - y'(0) = s^2Y(s) - 7s - 5\,. \end{aligned}$$

The Laplace transform of the initial-value problem thereby becomes

$$(s^2+4)Y(s) - 7s - 5 = F(s)\,, \qquad \Longrightarrow \qquad Y(s) = \frac{1}{s^2+4}\big(7s + 5 + F(s)\big)\,.$$

All that remains to be done is to compute $F(s)$. The first step is to use unit step functions to express $f(t)$ in the form (6.16) as

$$\begin{aligned} f(t) &= \big(u(t) - u(t-2)\big)\,t^2 + \big(u(t-2) - u(t-4)\big)\,2t + u(t-4)\,4 \\ &= t^2 + u(t-2)\,(2t - t^2) + u(t-4)\,(4-2t)\,. \end{aligned}$$

The second step is to write this in form (6.17) as

$$f(t) = t^2 + u(t-2)\,j_1(t-2) + u(t-4)\,j_2(t-4)\,,$$

where

$$\begin{aligned} j_1(t) &= 2(t+2) - (t+2)^2 = 2t + 4 - t^2 - 4t - 4 = -t^2 - 2t\,, \\ j_2(t) &= 4 - 2(t+4) = -2t - 4\,. \end{aligned}$$

Here we obtained $j_1(t)$ by replacing t with $t+2$ in the factor $(2t - t^2)$ and $j_2(t)$ by replacing t with $t+4$ in the factor $(4-2t)$. Finally, the above form for $f(t)$ allows us to use the sixth entry of table (**??**) to compute $F(s) = \mathcal{L}[f](s)$ in the form (6.18) as

$$\begin{aligned} F(s) &= \mathcal{L}[t^2](s) + \mathcal{L}[u(t-2)\,j_1(t-2)](s) + \mathcal{L}[u(t-4)\,j_2(t-4)](s) \\ &= \mathcal{L}[t^2](s) - e^{-2s}\mathcal{L}[t^2 + 2t](s) - e^{-4s}\mathcal{L}[2t+4](s) \\ &= \frac{2}{s^3} - e^{-2s}\left(\frac{2}{s^3} + \frac{2}{s^2}\right) - e^{-4s}\left(\frac{2}{s^2} + \frac{4}{s}\right) \\ &= \big(1 - e^{-2s}\big)\,\frac{2}{s^3} - \big(e^{-2s} + e^{-4s}\big)\,\frac{2}{s^2} - e^{-4s}\,\frac{4}{s}\,. \end{aligned}$$

It follows that

$$Y(s) = \frac{7s+5}{s^2+4} + \big(1 - e^{-2s}\big)\,\frac{2}{s^3(s^2+4)} - \big(e^{-2s} + e^{-4s}\big)\,\frac{2}{s^2(s^2+4)} - e^{-4s}\,\frac{4}{s\,(s^2+4)}\,.$$

6.7. **Inverse Transform.** The process of determining $y(t)$ from $Y(s)$ is called taking the inverse Laplace transform. It is important to know that this process has a unique result. Indeed, we will use the following theorem.

Theorem. Let $f(t)$ and $g(t)$ be two functions over $[0, \infty)$ and α a real number such that

- $f(t)$ and $g(t)$ are of exponential order α as $t \to \infty$,
- $f(t)$ and $g(t)$ are piecewise continuous over every $[0, T]$,
- $\mathcal{L}[f](s) = \mathcal{L}[g](s)$ for every $s > \alpha$.

Then $f(t) = g(t)$ for every t in $[0, \infty)$.

The proof of this result requires tools from complex variables that are beyond the scope of this course. Fortunately, you do not need to know how to prove this result to use it! Its usefulness stems from the fact that solutions $y(t)$ to the initial-value problems we are considering lie within the class of functions considered above — namely, they are functions that are of exponential order as $t \to \infty$ and that are piecewise continuous over every $[0, T]$. In fact, they are continuous and piecewise differentiable over every $[0, T]$. This means that if we succeed in finding a function $y(t)$ within this class such that $\mathcal{L}[y](s) = Y(s)$ then it will be the unique solution of the initial-value problem that we seek.

Because the above result states there is a unique $f(t)$ that is of exponential order as $t \to \infty$ and is piecewise continuous over every $[0, T]$ such that $\mathcal{L}[f](s) = F(s)$, we introduce the notation

$$f(t) = \mathcal{L}^{-1}[F](t) .$$

The operator $\mathcal{L}^{-1}$ denotes the *inverse Laplace transform.* Because it undoes the Laplace transform $\mathcal{L}$, it inherits many properties from $\mathcal{L}$. For example, it is linear. We can also easily read-off from the first and last three entries in table (6.14) of basic forms that

$$\begin{aligned} \mathcal{L}^{-1}\left[\frac{n!}{s^{n+1}}\right](t) &= t^n , & \mathcal{L}^{-1}\left[\frac{n!}{(s-a)^{n+1}}\right](t) &= e^{at} t^n , \\ \mathcal{L}^{-1}\left[\frac{s}{s^2+b^2}\right](t) &= \cos(bt) , & \mathcal{L}^{-1}\left[\frac{s-a}{(s-a)^2+b^2}\right](t) &= e^{at}\cos(bt) , \\ \mathcal{L}^{-1}\left[\frac{b}{s^2+b^2}\right](t) &= \sin(bt) , & \mathcal{L}^{-1}\left[\frac{b}{(s-a)^2+b^2}\right](t) &= e^{at}\sin(bt) . \end{aligned} \tag{6.19}$$

It is also clear from the sixth entry of table (6.14) that

$$\mathcal{L}^{-1}[e^{-cs} J(s)](t) = u(t-c)\, j(t-c) , \quad \text{where } j(t) = \mathcal{L}^{-1}[J](t) . \tag{6.20}$$

For us, the process of computing $y(t) = \mathcal{L}^{-1}[Y](t)$ for a given $Y(s)$ will be one of expressing $Y(s)$ as a sum of terms that will allow us to read off $y(t)$ from the basic forms above. To illustrate this process, we will compute $y(t) = \mathcal{L}^{-1}[Y](t)$ for the $Y(s)$ found in the examples given in the previous section, thereby completing our solution of the initial-value problems.

Example. Find $y(t) = \mathcal{L}^{-1}[Y](t)$ for

$$Y(s) = \frac{1}{(s-2)(s-5)} + \frac{3}{s-2} .$$

Solution. By the partial fraction identity

$$\frac{1}{(s-2)(s-5)} = \frac{\frac{1}{3}}{s-5} + \frac{-\frac{1}{3}}{s-2}\,,$$

we can express $Y(s)$ as

$$Y(s) = \tfrac{1}{3}\frac{1}{s-5} + \tfrac{8}{3}\frac{1}{s-2}\,.$$

The top right entry of table (6.19) with $a = 5$ and $a = 2$ then yields

$$y(t) = \mathcal{L}^{-1}[Y(s)](t) = \tfrac{1}{3}\mathcal{L}^{-1}\bigg[\frac{1}{s-5}\bigg](t) + \tfrac{8}{3}\mathcal{L}^{-1}\bigg[\frac{1}{s-2}\bigg](t) = \tfrac{1}{3}e^{5t} + \tfrac{8}{3}e^{2t}\,.$$

Example. Find $y(t) = \mathcal{L}^{-1}[Y](t)$ for

$$Y(s) = \frac{3s+1}{s^2-2s-8}\,.$$

Solution. By the partial fraction identity

$$\frac{3s+1}{s^2-2s-8} = \frac{3s+1}{(s-4)(s+2)} = \frac{\frac{13}{6}}{s-4} + \frac{\frac{5}{6}}{s+2}\,,$$

we can express $Y(s)$ as

$$Y(s) = \tfrac{13}{6}\frac{1}{s-4} + \tfrac{5}{6}\frac{1}{s+2}\,.$$

The top right entry of table (6.19) with $a = 4$ and $a = -2$ then yields

$$y(t) = \mathcal{L}^{-1}[Y(s)](t) = \tfrac{13}{6}\mathcal{L}^{-1}\bigg[\frac{1}{s-4}\bigg](t) + \tfrac{5}{6}\mathcal{L}^{-1}\bigg[\frac{1}{s+2}\bigg](t) = \tfrac{13}{6}e^{4t} + \tfrac{5}{6}e^{-2t}\,.$$

Example. Find $y(t) = \mathcal{L}^{-1}[Y](t)$ for

$$Y(s) = \frac{3}{(s^2+4)(s^2+9)}\,.$$

Solution. By the partial fraction identity

$$\frac{3}{(z+4)(z+9)} = \frac{\frac{3}{5}}{(z+4)} + \frac{-\frac{3}{5}}{(z+9)}\,,$$

we can express $Y(s)$ as

$$Y(s) = \frac{\frac{3}{5}}{s^2+4} - \frac{\frac{3}{5}}{s^2+9} = \tfrac{3}{10}\frac{2}{s^2+2^2} - \tfrac{1}{5}\frac{3}{s^2+3^2}\,.$$

The bottom left entry of table (6.19) with $b = 2$ and $b = 3$ then yields

$$y(t) = \mathcal{L}^{-1}[Y(s)](t) = \tfrac{3}{10}\mathcal{L}^{-1}\bigg[\frac{2}{s^2+2^2}\bigg](t) - \tfrac{1}{5}\mathcal{L}^{-1}\bigg[\frac{3}{s^2+3^2}\bigg](t) = \tfrac{3}{10}\sin(2t) - \tfrac{1}{5}\sin(3t)\,.$$

Example. Find $y(t) = \mathcal{L}^{-1}[Y](t)$ for

$$Y(s) = \frac{7s+5}{s^2+4} + \big(1-e^{-2s}\big)\frac{2}{s^3(s^2+4)} - \big(e^{-2s}+e^{-4s}\big)\frac{2}{s^2(s^2+4)} - e^{-4s}\frac{4}{s\,(s^2+4)}\,.$$

Solution. We first derive the partial fraction identities

$$\frac{7s+5}{s^2+4} = \frac{7s}{s^2+4} + \frac{5}{s^2+4}\,, \qquad \frac{2}{s^2(s^2+4)} = \frac{\frac{1}{2}}{s^2} - \frac{\frac{1}{2}}{s^2+4}\,,$$
$$\frac{2}{s^3(s^2+4)} = \frac{\frac{1}{2}}{s^3} - \frac{\frac{1}{8}}{s} + \frac{\frac{1}{8}s}{s^2+4}\,, \qquad \frac{4}{s\,(s^2+4)} = \frac{1}{s} - \frac{s}{s^2+4}\,.$$

The top left identity is straightforward. The top right identity only involves s^2, so it is simply the identity

$$\frac{2}{z(z+4)} = \frac{\frac{1}{2}}{z} - \frac{\frac{1}{2}}{z+4}\,, \quad \text{evaluated at } z = s^2\,.$$

The bottom right identity is simply $2s$ times the top right one. Finally, the bottom left identity is obtained by first dividing the top right one by s and then employing the bottom right one divided by 8 to the last term.

These partial fraction identites allow us to express $Y(s)$ as

$$\begin{aligned} Y(s) &= \frac{7s}{s^2+4} + \frac{5}{s^2+4} + \left(1 - e^{-2s}\right)\left(\frac{\frac{1}{2}}{s^3} - \frac{\frac{1}{8}}{s} + \frac{\frac{1}{8}s}{s^2+4}\right) \\ &\quad - \left(e^{-2s} + e^{-4s}\right)\left(\frac{\frac{1}{2}}{s^2} - \frac{\frac{1}{2}}{s^2+4}\right) - e^{-4s}\left(\frac{1}{s} - \frac{s}{s^2+4}\right) \\ &= 7\frac{s}{s^2+2^2} + \frac{5}{2}\frac{2}{s^2+2^2} + \left(1 - e^{-2s}\right)\left(\frac{1}{4}\frac{2}{s^3} - \frac{1}{8}\frac{1}{s} + \frac{1}{8}\frac{s}{s^2+2^2}\right) \\ &\quad - \left(e^{-2s} + e^{-4s}\right)\left(\frac{1}{2}\frac{1}{s^2} - \frac{1}{4}\frac{2}{s^2+2^2}\right) - e^{-4s}\left(\frac{1}{s} - \frac{s}{s^2+2^2}\right). \end{aligned}$$

The formulas in the first column of table (6.19) show that

$$\begin{aligned} \mathcal{L}^{-1}\left[7\frac{s}{s^2+2^2} + \frac{5}{2}\frac{2}{s^2+2^2}\right](t) &= 7\cos(2t) + \tfrac{5}{2}\cos(2t)\,, \\ \mathcal{L}^{-1}\left[\frac{1}{4}\frac{2}{s^3} - \frac{1}{8}\frac{1}{s} + \frac{1}{8}\frac{s}{s^2+2^2}\right](t) &= \tfrac{1}{4}t^2 - \tfrac{1}{8} + \tfrac{1}{8}\cos(2t)\,, \\ \mathcal{L}^{-1}\left[\frac{1}{2}\frac{1}{s^2} - \frac{1}{4}\frac{2}{s^2+2^2}\right](t) &= \tfrac{1}{2}t - \tfrac{1}{4}\sin(2t)\,, \\ \mathcal{L}^{-1}\left[\frac{1}{s} - \frac{s}{s^2+2^2}\right](t) &= 1 - \cos(2t)\,. \end{aligned}$$

By combining these facts with formula (6.20), it follows that

$$\begin{aligned} y(t) &= 7\cos(2t) + \tfrac{5}{2}\cos(2t) + \left(\tfrac{1}{4}t^2 - \tfrac{1}{8} + \tfrac{1}{8}\cos(2t)\right) \\ &\quad - u(t-2)\left(\tfrac{1}{4}(t-2)^2 - \tfrac{1}{8} + \tfrac{1}{8}\cos(2(t-2))\right) \\ &\quad - u(t-2)\left(\tfrac{1}{2}(t-2) - \tfrac{1}{4}\sin(2(t-2))\right) \\ &\quad - u(t-4)\left(\tfrac{1}{2}(t-4) - \tfrac{1}{4}\sin(2(t-4))\right) \\ &\quad - u(t-4)\left(1 - \cos(2(t-4))\right). \end{aligned}$$

6.8. **Computing Green Functions.** The Laplace transform can be used to efficiently compute Green functions for differential operators with constant coefficients. Recall that given the n^{th}-order differential operator L with constant coefficients given by

$$\mathrm{L} = \mathrm{D}^n + a_1 \mathrm{D}^{n-1} + \cdots + a_{n-1}\mathrm{D} + a_n \,,$$

the Green function $g(t)$ associated with L is the solution of the initial-value problem

$$\begin{gathered} g^{(n)} + a_1 g^{(n-1)} + \cdots + a_{n-1} g' + a_n g = 0 \,, \\ g(0) = 0 \,, \quad g'(0) = 0 \,, \quad \cdots \quad g^{(n-2)}(0) = 0 \,, \quad g^{(n-1)}(0) = 1 \,. \end{gathered}$$

The Laplace transform of this initial-value problem is

$$\mathcal{L}\big[g^{(n)}\big](s) + a_1 \mathcal{L}\big[g^{(n-1)}\big](s) + \cdots + a_{n-1}\mathcal{L}[g'](s) + \mathcal{L}[g](s) = 0 \,,$$

where if $G(s) = \mathcal{L}[g](s)$ then

$$\begin{aligned} \mathcal{L}[g'](s) &= s\,\mathcal{L}[g](s) - g(0) = s\,G(s) \,, \\ \mathcal{L}\big[g''\big](s) &= s\,\mathcal{L}[g'](s) - g'(0) = s^2 G(s) \,, \\ &\vdots \\ \mathcal{L}\big[g^{(n-1)}\big](s) &= s\,\mathcal{L}[g^{(n-2)}](s) - g^{(n-2)}(0) = s^{n-1} G(s) \,, \\ \mathcal{L}\big[g^{(n)}\big](s) &= s\,\mathcal{L}[g^{(n-1)}](s) - g^{(n-1)}(0) = s^n G(s) - 1 \,. \end{aligned}$$

We thereby see that $G(s)$ satisfies

$$p(s)G(s) - 1 = 0 \,,$$

where $p(s)$ is the characteristic polynomial of L, which is given by

$$p(s) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n \,.$$

Therefore $G(s)$ is given by

$$G(s) = \frac{1}{p(s)} \,, \tag{6.21}$$

In other words, the Laplace transform of the Green function of L is the reciprocal of the characteristic polynomial of L.

The problem of computing a Green function is thereby reduced to the problem of finding an inverse Laplace transform. This can often be done quickly.

Example. Find the Green function $g(t)$ for the operator $\mathrm{L} = \mathrm{D}^2 + 6\mathrm{D} + 13$.

Solution. Because $p(s) = s^2 + 6s + 13 = (s+3)^2 + 2^2$, the bottom right entry of table (6.19) with $a = -3$ and $b = 2$ and formula (6.21) shows that the Green function is given by

$$g(t) = \mathcal{L}^{-1}\bigg[\frac{1}{p(s)}\bigg] = \frac{1}{2}\mathcal{L}^{-1}\bigg[\frac{2}{(s+3)^2 + 2^2}\bigg] = \frac{1}{2}e^{-3t}\sin(2t) \,.$$

Example. Find the Green function $g(t)$ for the operator $\mathrm{L} = \mathrm{D}^2 + 2\mathrm{D} - 15$.

Solution. Because $p(s) = s^2 + 2s - 15 = (s-3)(s+5)$, we use the partial fraction identity

$$\frac{1}{(s-3)(s+5)} = \frac{\frac{1}{8}}{s-3} - \frac{\frac{1}{8}}{s+5}\,.$$

The top right entry of table (6.19) with $a = 3$ and with $a = -5$ and formula (6.21) shows that the Green function is given by

$$g(t) = \mathcal{L}^{-1}\left[\frac{1}{p(s)}\right] = \frac{1}{8}\mathcal{L}^{-1}\left[\frac{1}{s-3}\right] - \frac{1}{8}\mathcal{L}^{-1}\left[\frac{1}{s+5}\right] = \frac{e^{3t} - e^{-5t}}{8}\,.$$

Example. Find the Green function $g(t)$ for the operator $\mathrm{L} = \mathrm{D}^4 + 13\mathrm{D}^2 + 36$.

Solution. Because $p(s) = s^4 + 13s^2 + 36 = (s^2+4)(s^2+9)$ only depends on s^2, we can use the partial fraction identity

$$\frac{1}{(z+4)(z+9)} = \frac{\frac{1}{5}}{z+4} - \frac{\frac{1}{5}}{z+9} \quad \text{at } z = s^2\,.$$

The bottom left entry of table (6.19) with $b = 2$ and with $b = 3$ and formula (6.21) shows that the Green function is given by

$$g(t) = \mathcal{L}^{-1}\left[\frac{1}{p(s)}\right] = \frac{1}{10}\mathcal{L}^{-1}\left[\frac{2}{s^2+2^2}\right] - \frac{1}{15}\mathcal{L}^{-1}\left[\frac{3}{s^2+3^2}\right] = \frac{\sin(2t)}{10} - \frac{\sin(3t)}{15}\,.$$

6.9. **Convolutions.** Let $f(t)$ and $g(t)$ be any two functions defined over the interval $[0,\infty)$. Their *convolution* is a third function $(f*g)(t)$ that is defined by the formula

$$(f*g)(t) = \int_0^t f(t-\tau)g(\tau)\,\mathrm{d}\tau\,, \tag{6.22}$$

whenever the above integral makes sense for every $t \geq 0$. This will be the case whenever both f and g are piecewise continuous over every $[0,T]$.

The convolution can be thought of a some kind of product between two functions. It is easily checked that this so-called convolution product satisfies some of the properties of ordinary multipulcation. For example, for any functions f, g, amd h that are piecewise continuous over every $[0,T]$ we have

$$\begin{aligned} g*f &= f*g && \text{commutative law}\,,\\ h*(f+g) &= h*f + h*g && \text{distributive law}\,,\\ h*(g*f) &= (h*g)*f && \text{associative law}\,. \end{aligned} \tag{6.23}$$

The commutative law is proved by introducing $\tau' = t-\tau$ as a new variable of integration, whereby we see that

$$(g*f)(t) = \int_0^t g(t-\tau)f(\tau)\,\mathrm{d}\tau = \int_0^t g(\tau')f(t-\tau')\,\mathrm{d}\tau' = (f*g)(t)\,.$$

Verification of the distributive and associative laws is left as an exercise.

The convolution differs from ordinary multipulcation in some respects too. For example, it is not generaly true that $f * 1 = f$ or that $f * f \geq 0$. Indeed, we see that

$$(1 * 1)(t) = \int_0^t 1 \cdot 1 \, \mathrm{d}\tau = t \neq 1 \,,$$

and that

$$\begin{aligned}(\sin * \sin)(t) &= \int_0^t \sin(t-\tau)\sin(\tau)\,\mathrm{d}\tau \\ &= \sin(t)\int_0^t \cos(\tau)\sin(\tau)\,\mathrm{d}\tau + \cos(t)\int_0^t \sin(\tau)^2\,\mathrm{d}\tau \\ &= \tfrac{1}{2}\sin(t)^3 + \tfrac{1}{2}t\cos(t) - \tfrac{1}{2}\sin(t)\cos(t)^2 \ngeq 0 \quad \text{for every } t > 0\,.\end{aligned}$$

In fact, we can show that $1 * f = f$ if and only if $f = 0$.

The main result of this section is that the Laplace transform of a convolution of two functions is the ordinary product of their Laplace transforms. In other words, the Laplace transform maps convolutions to multiplication.

Convolution Theorem. Let $f(t)$ and $g(t)$ be

- piecewise continuous over every $[0, T]$
- of exponential order α as $t \to \infty$.

Then $f * g(t)$ is continuous over $[0, \infty)$ and is of exponential order α as $t \to \infty$. Moreover, $\mathcal{L}[f * g](s)$ is defined for every $s > \alpha$ with

$$\mathcal{L}[f * g](s) = F(s)G(s)\,, \quad \text{where } F(s) = \mathcal{L}[f](s) \text{ and } G(s) = \mathcal{L}[g](s)\,. \tag{6.24}$$

Proof of (6.24). For every $T > 0$ definition (6.22) of convolution implies that

$$\int_0^T e^{-st}\,(f * g)(t)\,\mathrm{d}t = \int_0^T e^{-st}\int_0^t f(t-\tau)g(\tau)\,\mathrm{d}\tau\,\mathrm{d}t = \int_0^T\int_0^t e^{-st} f(t-\tau)g(\tau)\,\mathrm{d}\tau\,\mathrm{d}t\,.$$

We now exchange the order of the definite integrals over τ and t on the right-hand side. As you recall from multivariable Calculus, this should be done carefully because the upper endpoint of the inner integral depends on the variable of integration t of the outer integral. When viewed in the (τ, t)-plane, the domain over which the double integral is being taken is the triangle given by $0 \leq \tau \leq t \leq T$. In general, when the order of definite integrals is exchanged over this domain we have

$$\int_0^T\int_0^t \bullet \;\mathrm{d}\tau\,\mathrm{d}t = \int_0^T\int_\tau^T \bullet \;\mathrm{d}t\,\mathrm{d}\tau\,,$$

where $\bullet$ denotes any appropriate integrand. We thereby obtain

$$\int_0^T e^{-st}\,(f * g)(t)\,\mathrm{d}t = \int_0^T\int_\tau^T e^{-st} f(t-\tau)g(\tau)\,\mathrm{d}t\,\mathrm{d}\tau\,.$$

We now factor e^{-st} as $e^{-st} = e^{-s(t-\tau)}e^{-s\tau}$, and group the factor $e^{-s(t-\tau)}$ with $f(t-\tau)$ and the factor $e^{-s\tau}$ with $g(\tau)$, whereby

$$\begin{aligned}\int_0^T e^{-st}\,(f*g)(t)\,\mathrm{d}t &= \int_0^T\int_\tau^T e^{-s(t-\tau)}f(t-\tau)\,e^{-s\tau}g(\tau)\,\mathrm{d}t\,\mathrm{d}\tau \\ &= \int_0^T e^{-s\tau}g(\tau)\int_\tau^T e^{-s(t-\tau)}f(t-\tau)\,\mathrm{d}t\,\mathrm{d}\tau\,.\end{aligned}$$

We then make the change of variable $t' = t-\tau$ in the inner definite integral to obtain

$$\int_0^T e^{-st}\,(f*g)(t)\,\mathrm{d}t = \int_0^T e^{-s\tau}g(\tau)\int_0^{T-\tau} e^{-st'}f(t')\,\mathrm{d}t'\,\mathrm{d}\tau\,.$$

Upon formally letting $T\to\infty$ above, definition (6.2) of the Laplace transform shows that the inner integral converges to $F(s)$, which is independent of τ. The double integral thereby converges to $G(s)F(s)$, yielding (6.24). □

Remark. Because the upper endpoint of the inner integral depends on the variable of integration τ of the outer integral, properly passing to the limit above requires greater care than we took here. The techniques we need are taught in Advanced Calculus courses. The argument given above suits our purposes because it illuminates why (6.24) holds.

The convolution theorem can be used to help evaluate inverse Laplace transforms. For example, suppose that we know for a given $F(s)$ and $G(s)$ that $f(t) = \mathcal{L}^{-1}[F](t)$ and $g(t) = \mathcal{L}^{-1}[G](t)$. Then (6.24) implies that

$$\mathcal{L}^{-1}\big[F(s)G(s)\big](t) = (f*g)(t)\,. \tag{6.25}$$

This fact can be used to express inverse Laplace transforms as convolutions. We may still have to evaluate the convolution integral, but some of people might find that easier than using partial fraction identities to express $F(s)G(s)$ in basic forms.

Example. Compute $y(t) = \mathcal{L}^{-1}[Y](t)$ for

$$Y(s) = \frac{2}{s^2(s^2+4)}\,.$$

Solution. Because we know from table (6.19) that

$$\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] = t\,,\qquad \mathcal{L}^{-1}\left[\frac{2}{s^2+2^2}\right] = \sin(2t)\,,$$

it follows from (6.25) and an integration by parts that

$$\begin{aligned}y(t) = \mathcal{L}^{-1}\left[\frac{2}{s^2(s^2+4)}\right] &= \mathcal{L}^{-1}\left[\frac{1}{s^2}\,\frac{2}{s^2+2^2}\right] = \int_0^t (t-\tau)\sin(2\tau)\,\mathrm{d}\tau \\ &= (\tau-t)\frac{\cos(2\tau)}{2}\bigg|_0^t - \int_0^t \frac{\cos(2\tau)}{2}\,\mathrm{d}\tau = \frac{t}{2} - \frac{\sin(2t)}{4}\,.\end{aligned}$$

This is the same result we got on page 18 using a partial fraction identity.

The convolution theorem gives us another way to understand Green functions. We have used the Green function to construct a particular solution of the nonhomogeneous equation $\mathrm{L}y = p(\mathrm{D}) = f(t)$ by the formula

$$y_P(t) = \int_0^t g(t-\tau) f(\tau)\,\mathrm{d}\tau\,.$$

Notice that the right-hand side above is exactly $(g * f)(t)$. Upon taking the Laplace transform of this formula, the Convolution Theorem and formula (6.21) then yield

$$\mathcal{L}[y_P](s) = \mathcal{L}[g*f](s) = G(s)F(s) = \frac{F(s)}{p(s)}\,, \qquad \text{where } F(s) = \mathcal{L}[f](s)\,.$$

But this agrees with formula (6.13) with $q(s) = 0$. Indeed, recall that $y_P(t)$ given by the Green function formula satisfies the initial conditions

$$y_P(0) = 0\,, \quad y_P'(0) = 0\,, \quad \cdots \quad y_P^{(n-2)}(0) = 0\,, \quad y_P^{(n-1)}(0) = 0\,.$$

Because $y_P(t)$ satisfies the initial-value problem (6.12) with $y_0 = y_1 = \cdots = y_{n-1} = 0$, the polynomial $q(s)$ appearing in (6.13) vanishes.

6.10. **Impulse Forcing.** Let us consider the family of piecewise-defined forcing functions given by

$$f_\tau(t) = \frac{1}{\tau}\big(u(t) - u(t-\tau)\big)\,, \qquad \text{for some } \tau > 0\,. \tag{6.26}$$

This forcing $f_\tau(t)$ has amplitude $1/\tau$ that turns on at $t = 0$ and turns off at $t = \tau$. The subscript τ indicates this dependence. We want to consider the effect of such a forcing on the solution of the initial-value problem

$$\mathrm{L}y = f_\tau(t)\,, \qquad y(0) = y'(0) = \cdots = y^{(n-1)}(0) = 0\,, \tag{6.27a}$$

where L is the n^{th}-order differential operator with constant coefficients given by

$$\mathrm{L} = \mathrm{D}^n + a_1\mathrm{D}^{n-1} + \cdots + a_{n-1}\mathrm{D} + a_n\,. \tag{6.27b}$$

More specifically, we want to understand the behavior of this solution when τ is very small — that is, when there is a strong force of short duration. Such a force is called an *impulse.*

The solution of the initial-value problem (6.27) will depend upon τ through $f_\tau(t)$, so we will denote it y_τ. Then the Laplace transform of the initial-value problem is

$$\mathcal{L}[y_\tau^{(n)}](s) + a_1\mathcal{L}[y_\tau^{(n-1)}](s) + \cdots + a_{n-1}\mathcal{L}[y_\tau'](s) + a_n\mathcal{L}[y_\tau](s) = F_\tau(s)\,,$$

where

$$\begin{aligned}
\mathcal{L}[y_\tau](s) &= Y_\tau(s)\,,\\
\mathcal{L}[y_\tau'](s) &= sY_\tau(s) - y_\tau(0) = sY_\tau(s)\,,\\
&\ \vdots\\
\mathcal{L}[y_\tau^{(n-1)}](s) &= s\,\mathcal{L}[y_\tau^{(n-2)} - y_\tau^{(n-2)}(0) = s^{n-1}Y_\tau(s)\,,\\
\mathcal{L}[y_\tau^{(n)}](s) &= s\,\mathcal{L}[y_\tau^{(n-1)} - y_\tau^{(n-1)}(0) = s^{n}Y_\tau(s)\,,
\end{aligned}$$

and

$$F_\tau(s) = \mathcal{L}[f_\tau](s) = \frac{1}{\tau}\big(\mathcal{L}[u](s) - \mathcal{L}[u(t-\tau)](s)\big) = \frac{1-e^{-\tau s}}{\tau s} \quad \text{for every } s > 0\,.$$

The Laplace transform of the initial-value problem thereby becomes

$$p(s)Y_\tau(s) = \frac{1-e^{-\tau s}}{\tau s}\,,$$

where $p(s)$ is the characteristic polynomial of L, which is given by

$$p(s) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n\,.$$

Therefore the Laplace transform $Y_\tau(s)$ of the solution $y_\tau(t)$ is given in terms of $F_\tau(s)$ as

$$Y_\tau(s) = \frac{1}{p(s)}\,\frac{1-e^{-\tau s}}{\tau s}\,. \tag{6.28}$$

Now let $Y(s)$ denote the limit of $Y_\tau(s)$ as τ becomes small. We see from (6.28) that

$$Y(s) = \lim_{\tau\to 0} Y_\tau(s) = \frac{1}{p(s)}\,\lim_{\tau\to 0}\frac{1-e^{-\tau s}}{\tau s} = \frac{1}{p(s)}\,,$$

where the last limit can be evaluated either by the l'Hospital rule or by making a Taylor approximation of the numerator. But $1/p(s)$ is the Laplace transform of the Green function $g(t)$ associated with the differential operator (6.27b). Therefore it seems as if the solution of the initial-value problem (6.27) will behave like the Green function $g(t)$ as τ becomes small.

If is natural to wonder if this result depends upon the particular form (6.26) of the forcing that we considered. To explore this question we now consider the initial-value problem (6.27) for the more general family of forcing functions given by

$$f_\tau(t) = \frac{1}{\tau}\,f\!\left(\frac{t}{\tau}\right)\,, \qquad \text{for some } \tau > 0\,, \tag{6.29}$$

where $f(t)$ is any nonnegative piecewise integrable function of exponential order $\alpha < 0$. (The forcing (6.26) has this form with $f(t) = u(t) - u(t-1)$.)

Because $\alpha < 0$ and $F(s) = \mathcal{L}[f](s)$ is defined for every $s > \alpha$, this implies that

$$F(0) = \int_0^\infty f(t)\,\mathrm{d}t < \infty\,.$$

Then

$$\begin{aligned} F_\tau(s) = \mathcal{L}[f_\tau](s) &= \int_0^\infty e^{-st} f_\tau(t)\,\mathrm{d}t = \frac{1}{\tau}\int_0^\infty e^{-st} f\!\left(\frac{t}{\tau}\right)\,\mathrm{d}t \\ &= \int_0^\infty e^{-\tau s t'} f(t')\,\mathrm{d}t' = F(\tau s) \quad \text{for every } s > \frac{\alpha}{\tau}\,. \end{aligned}$$

Notice that this is consistent with formula (6.28) that was derived for our original forcing (6.26).

The Laplace transform $Y_\tau(s)$ of the solution $y_\tau(t)$ to the initial-value problem (6.27) with this general forcing is given in terms of $F_\tau(s)$ as

$$Y_\tau(s) = \frac{1}{p(s)}\, F_\tau(s) = \frac{1}{p(s)}\, F(\tau s)\,. \tag{6.30}$$

Again let $Y(s)$ denote the limit of $Y_\tau(s)$ as τ becomes small. We see from (6.30) that

$$Y(s) = \lim_{\tau\to 0} Y_\tau(s) = \frac{1}{p(s)} \lim_{\tau\to 0} F(\tau s) = \frac{1}{p(s)}\, F(0) = \frac{1}{p(s)} \int_0^\infty f(t')\,\mathrm{d}t'\,.$$

Because $1/p(s)$ is the Laplace transform of the Green function $g(t)$, it seems that $y_\tau(t)$ behaves like a multiple of the Green function as τ becomes small. Specifically, it seems that

$$y_\tau(t) \approx g(t) \int_0^\infty f(t')\,\mathrm{d}t' \quad \text{for small } \tau\,.$$

This shows that the details of an impulse do not matter. All that matters is the integral of an impulse forcing.

Therefore for sufficiently small τ every forcing $f_\tau(t)$ given by (6.29) for some $f(t)$ can be modeled by an idealized impulse forcing $M\delta(t)$, where M is the magnitude of the impulse, which is given by

$$M = \int_0^\infty f(t)\,\mathrm{d}t\,,$$

and $\delta(t)$ is commonly called either the *unit impulse* or *Dirac delta* function even though it is *not a function*! For every interval $[a,b]$ such that $0 \in [a,b]$ the unit impulse is treated like it has the property

$$\int_a^b \delta(t)\,\phi(t)\,\mathrm{d}t = \phi(0) \quad \text{for every } \phi \text{ that is continuous over } [a,b]\,. \tag{6.31a}$$

For every interval $[a,b]$ such that $0 \notin [a,b]$ the unit impulse is treated like it has the property

$$\int_a^b \delta(t)\,\phi(t)\,\mathrm{d}t = 0 \quad \text{for every } \phi \text{ that is continuous over } [a,b]\,. \tag{6.31b}$$

Remark. The notion of a unit impulse goes back to Oliver Heaviside in 1899. Paul Dirac introduced the delta notation in his work on quantum mechanics during the late 1920s. These early approaches raised as many questions as they answered. Subsequently Soloman Bochner (1933), Sergei Sobolev (1938), and Kurt Friedrichs (1944), took important steps towards putting the notion on a firmer mathematical footing. Laurant Schwartz developed the theory of *distributions* in the 1950s, which provides a framwork in which the impulse function and other so-called *generalized functions* exist. These theories lie far beyond the scope of this course. Our motivation comes from Heaviside, our notation comes from Dirac, and our property (6.31) comes from Schwartz.

For every ϕ that is continuous over $[a,b]$ the shift of the unit impulse function $\delta(t-c)$ is treated like

$$\int_a^b \delta(t-c)\,\phi(t)\,\mathrm{d}t = \int_{a-c}^{b-c} \delta(t)\,\phi(t+c)\,\mathrm{d}t\,. \tag{6.32}$$

Therefore for every ϕ that is continuous over $[a,b]$ we have

$$\int_a^b \delta(t-c)\,\phi(t)\,\mathrm{d}t = \begin{cases} \phi(c) & \text{if } c \in [a,b]\,, \\ 0 & \text{if } c \notin [a,b]\,. \end{cases}$$

These properties allow us to compute

$$\mathcal{L}[\delta](s) = \int_0^\infty e^{-st}\delta(t)\,\mathrm{d}t = 1\,,$$
$$\mathcal{L}[\delta(t-c)](s) = \int_0^\infty e^{-st}\delta(t-c)\,\mathrm{d}t = e^{-cs} \quad \text{for every } c > 0\,.$$

Therefore the initial-value problem (6.27) with $f(t) = M\delta(t)$ has solution $y(t) = Mg(t)$.

Example. Solve the initial-value problem

$$y''' - 4y'' + 3y' = 5\delta(t-2)\,, \qquad y(0) = y'(0) = 0\,,\ y''(0) = 7\,.$$

Solution. The Laplace transform of the initial-value problem is

$$\mathcal{L}[y'''](s) - 4\mathcal{L}[y''](s) + 3\mathcal{L}[y'](s) = 5\mathcal{L}[\delta(t-2)](s)\,,$$

where

$$\mathcal{L}[y](s) = Y(s)\,,$$
$$\mathcal{L}[y'](s) = s\mathcal{L}[y](s) - y(0) = sY(s)\,,$$
$$\mathcal{L}[y''](s) = s\mathcal{L}[y'](s) - y'(0) = s^2Y(s)\,,$$
$$\mathcal{L}[y'''](s) = s\mathcal{L}[y''](s) - y''(s) = s^3Y(s) - 7\,,$$

and by property (6.32) we have

$$\mathcal{L}[\delta(t-2)](s) = \int_0^\infty e^{-st}\delta(t-2)\,\mathrm{d}t = e^{-2s}\,.$$

The Laplace transform of the initial-value problem thereby becomes

$$(s^3 - 4s^2 + 3s)Y(s) = 7 + 5e^{-2s}\,.$$

Therefore the Laplace transform of the solution is

$$Y(s) = \frac{7 + 5e^{-2s}}{s^3 - 4s^2 + 3s}\,.$$

By the partial fraction identity

$$\frac{1}{s^3 - 4s^2 + 3s} = \frac{1}{s(s-1)(s-3)} = \frac{\frac{1}{3}}{s} - \frac{\frac{1}{2}}{s-1} + \frac{\frac{1}{6}}{s-3}\,,$$

we see that

$$Y(s) = \left(\frac{\frac{7}{3}}{s} - \frac{\frac{7}{2}}{s-1} + \frac{\frac{7}{6}}{s-3}\right) + e^{-2s}\left(\frac{\frac{5}{3}}{s} - \frac{\frac{5}{2}}{s-1} + \frac{\frac{5}{6}}{s-3}\right)\,.$$

By taking the inverse Laplace transform we find that the solution is

$$y(t) = \left(\tfrac{7}{3} - \tfrac{7}{2}e^t + \tfrac{7}{6}e^{3t}\right) + u(t-2)\left(\tfrac{5}{3} - \tfrac{5}{2}e^{t-2} + \tfrac{5}{6}e^{3(t-2)}\right)\,.$$

6.11. **Green Functions and Natural Fundamental Sets.** The initial-value problem (6.12) is

$$\text{(6.33a)} \qquad \mathrm{L}y = f(t)\,, \qquad y(0) = y_0\,, \quad y'(0) = y_1\,, \quad \cdots \quad , y^{(n-1)}(0) = y_{n-1}\,,$$

where

$$\text{(6.33b)} \qquad \mathrm{L} = \mathrm{D}^n + a_1\mathrm{D}^{n-1} + \cdots + a_{n-1}\mathrm{D} + a_n\,.$$

Formula (6.13) for the solution of initial-value problem can be recast as

$$\text{(6.34a)} \qquad y(t) = y_H(t) + y_P(t)\,,$$

where

$$\text{(6.34b)} \qquad y_H(t) = \mathcal{L}^{-1}\left[\frac{q(s)}{p(s)}\right](t)\,, \qquad y_P(t) = \mathcal{L}^{-1}\left[\frac{F(s)}{p(s)}\right](t)\,,$$

Here $p(s)$ is the characterisitc polynomial of L, $q(s)$ is the polynomial given in terms of the initial data by

$$\text{(6.35)} \qquad \begin{aligned} q(s) = {} & \big(s^{n-1} + a_1 s^{n-2} + \cdots + a_{n-3}s^2 + a_{n-2}s + a_{n-1}\big)y_0 \\ & + \big(s^{n-2} + a_1 s^{n-3} + \cdots + a_{n-3}s + a_{n-2}\big)y_1 \\ & \vdots \\ & + \big(s^2 + a_1 s + a_2\big)y_{n-3} + \big(s + a_1\big)y_{n-2} + y_{n-1}\,, \end{aligned}$$

and $F(s) = \mathcal{L}[f](s)$ is the Laplace transform of the forcing $f(t)$. In this section we show how the decomposition of $y(t)$ given by (6.34) can be expressed in terms of the Green function $g(t)$ for the differential operator L.

The Convolution Theorem and formula (6.21) imply that

$$y_P(t) = \mathcal{L}^{-1}\left[\frac{F(s)}{p(s)}\right](t) = \mathcal{L}^{-1}\left[\frac{1}{p(s)}\right] * \mathcal{L}^{-1}[F](t) = g * f(t)\,.$$

This is the particular solution of $\mathrm{L}y = f(t)$ whose initial data are zero.

It follows that $y_H(t)$ is the solution of $\mathrm{L}y = 0$ whose initial data agree with $y(t)$. Because $G(s) = 1/p(s)$ by formula (6.21), we have

$$\mathrm{D}^k g(t) = \mathcal{L}^{-1}[s^k G(s)](t) = \mathcal{L}^{-1}[s^k/p(s)](t) \quad \text{for every } k = 0,\, 2,\, \cdots,\, n-1\,.$$

The function $y_H(t) = \mathcal{L}^{-1}[q(s)/p(s)](t)$ can thereby be expressed in terms of $g(t)$ by using (6.35) as

$$\begin{aligned} y_H(t) = {} & y_0\Big(\mathrm{D}^{n-1} + a_1\mathrm{D}^{n-2} + \cdots + a_{n-3}\mathrm{D}^2 + a_{n-2}\mathrm{D} + a_{n-1}\Big)g(t) \\ & + y_1\Big(\mathrm{D}^{n-2} + a_1\mathrm{D}^{n-3} + \cdots + a_{n-3}\mathrm{D} + a_{n-2}\Big)g(t) \\ & \vdots \\ & + y_{n-3}\Big(\mathrm{D}^2 + a_1\mathrm{D} + a_2\Big)g(t) + y_{n-2}\big(\mathrm{D} + a_1\big)g(t) + y_{n-1}g(t)\,. \end{aligned}$$

Therefore the *natural fundamental set of solutions* associated with the homogeneous equation $\mathrm{L}y = 0$ is given in terms of the Green function g by

$$
\begin{aligned}
N_0(t) &= \Big(\mathrm{D}^{n-1} + a_1\mathrm{D}^{n-2} + \cdots + a_{n-3}\mathrm{D}^2 + a_{n-2}\mathrm{D} + a_{n-1}\Big)g(t)\,,\\
N_1(t) &= \Big(\mathrm{D}^{n-2} + a_1\mathrm{D}^{n-3} + \cdots + a_{n-3}\mathrm{D} + a_{n-2}\Big)g(t)\,,\\
&\;\;\vdots\\
N_{n-3}(t) &= \Big(\mathrm{D}^2 + a_1\mathrm{D} + a_2\Big)g(t)\,,\\
N_{n-2}(t) &= \big(\mathrm{D} + a_1\big)g(t)\,,\\
N_{n-1}(t) &= g(t)\,.
\end{aligned}
\tag{6.36}
$$

The solution (6.34) of the initial-value problem (6.33) then can be expressed as

$$
y(t) = y_0N_0(t) + y_1N_1(t) + \cdots + y_{n-2}N_{n-2}(t) + y_{n-1}N_{n-1}(t) + (N_{n-1} * f)(t)\,,
$$

where $N_0(t)$, $N_1(t)$, $\cdots$, $N_{n-1}(t)$ is the natural fundamental set of solutions to the associated homogeneous equation, which is given in terms of the Green function $g(t)$ by (6.36). Recall that $W[N_0, N_1, \cdots, N_{n-1}](0) = 1$, which implies $W[N_0, N_1, \cdots, N_{n-1}](t) = e^{-a_1t}$ by the Abel formula.