UCLA Math 135, Winter 2015
Ordinary Differential Equations

7. Theory for First-Order Equations

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Contents

Return to Main Webpage

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7.1. Well-Posed Initial-Value Problems.

7.1.1. *Notion of Well-Posedness.* The notion of a well-posed problem is central to science and engineering. It motivated by the idea that mathematical problems in science and engineering are used to predict or explain something. A problem is called *well-posed* if

(i) the problem has a solution,

(ii) the solution is unique,

(iii) the solution depends continuously upon all the parameters in the problem.

The motivations for the first two points are fairly clear: a problem with no solution will not give a prediction, and a problem with many solutions gives too many. The third point is crucial. It recognizes that mathematical problems are models of reality. Nearby models are equally valid. To have predictive value, the solutions of nearby models should lie close to each other. For example, if our model is an initial-value problem associated with a differential equation then its solution should not change much if the initial value is a bit different or if a coefficient in the differential equation is altered slightly. This is what is meant by saying the solution depends continuously upon the problem.

The solution of a well-posed problem can be approximated accurately by a wealth of techniques. The solution of a problem that is not well-posed is very difficult, if not impossible to approximate accurately. This is why scientists and engineers want to know which problems are well-posed and which are not.

7.1.2. *Classical Solutions of Initial-Value Problems.* In this chapter we consider initial-value problems that can be put into the normal form

$$y' = f(t, y)\,, \qquad y(t_I) = y_I\,. \tag{7.1}$$

Notice that an initial-value problem consists of both a differential equation *and* an initial condition. Recall that a differential equation by itself will not have a unique solution.

Remark. We will often use t as the independent variable in our differential equations because in many applications the independent variable is time. However, the independent variable in a differential equation need not be time. Similarly, the dependent variable in a differential equation need not be y. Be prepared for the roles of independent and dependent variables in differential equations to be filled by many different letters!

In order to understand what is meant by a solution of the initial-value problem (7.1), we must understand what is meant by a solution of the differential equation in (7.1). The *classical* notion of solution is the following.

Definition 7.1. *If $Y(t)$ is a function defined for every t in an interval (a,b) then we say $Y(t)$ is a* solution *of the differential equation in (7.1) over (a,b) if*

$$
\begin{aligned}
&\textit{(i)} && Y'(t) \text{ is defined for every } t \text{ in } (a,b)\,,\\
&\textit{(ii)} && f(t,Y(t)) \text{ is defined for every } t \text{ in } (a,b)\,,\\
&\textit{(iii)} && Y'(t)=f(t,Y(t)) \text{ for every } t \text{ in } (a,b)\,.
\end{aligned}
\tag{7.2}
$$

We say $Y(t)$ is a solution *of the initial-value problem (7.1) over (a,b) if it is a solution of the differential equation that also satisfies the initial condition.*

Remark. We can recast condition (i) as "the function Y is differentiable over (a,b)." This definition is very natural in that it simply states (i) the thing on left-hand side of the equation makes sense, (ii) the thing on right-hand side of the equation makes sense, and (iii) the two things are equal. This classical notion of solution will suit our needs now. We will soon see that situations arise in which we will want to broaden this notion of solution.

In this chapter we will give conditions on $f(t,y)$ that insure the initial-value problem (7.1) has a unique classical solution. This theory underpins many of the graphical and numerical methods by which the solutions of these problems can be studied when analytical methods either do not apply or become complicated.

7.2. **Linear Equations.** We begin by specializing to *linear* differential equations. The normal form for these initial-value problems is

$$
y' + a(t)y = f(t)\,, \qquad y(t_I) = y_I\,. \tag{7.3}
$$

These problems can be put into the normal form (7.1) by rewriting the differential equation as

$$
y' = f(t) - a(t)y\,.
$$

They are among the simplest of all intial-value problems.

7.2.1. *Existence and Uniqueness Theorem.* The classical existence and uniqueness result for initial-value problem (7.3) is the following.

Theorem 7.1. *Let $a(t)$ and $f(t)$ be functions defined over the open interval (t_L,t_R) that are also continuous over (t_L,t_R).*

Then for every inital time t_I in (t_L,t_R), and every initial value y_I there exists a unique solution $y=Y(t)$ to initial-value problem (7.3) that is defined over (t_L,t_R).

Moreover, this solution is continuously differentiable and is given by the explicit formula

$$
y = \exp\left(-\int_{t_I}^{t} a(r)\,\mathrm{d}r\right) y_I + \int_{t_I}^{t} \exp\left(-\int_{s}^{t} a(r)\,\mathrm{d}r\right) f(s)\,\mathrm{d}s\,. \tag{7.4}
$$

Proof. We begin by introducing two functions that play a leading role in our proof. Because $a(t)$ and $f(t)$ are continuous over (t_L,t_R) and $t_I \in (t_L,t_R)$, we can define functions $A(t)$ and $B(t)$ over (t_L,t_R) by

$$
A(t) = \int_{t_I}^{t} a(r)\,\mathrm{d}r\,, \qquad B(t) = \int_{t_I}^{t} e^{A(s)} f(s)\,\mathrm{d}s\,. \tag{7.5}
$$

The Second Fundamental Theorem of Calculus states that these functions are differentiable at every t in (t_L, t_R) where their integrands are continuous, in which case

$$A'(t) = a(t)\,, \qquad B'(t) = e^{A(t)} f(t)\,. \tag{7.6}$$

But we have assumed that $a(t)$ is continuous over (t_L, t_R), so that $A(t)$ is continuously differentiable over (t_L, t_R) with $A'(t) = a(t)$. Because $A(t)$ is continuously differentiable over (t_L, t_R) and we have assumed that $f(t)$ is continuous over (t_L, t_R), we know that $e^{A(t)} f(t)$ is continuous over (t_L, t_R). It follows that $B(t)$ is continuously differentiable over (t_L, t_R) with $B'(t) = e^{A(t)} f(t)$.

We now verify that formula (7.4) indeed gives a solution of the initial-value problem (7.3). We use (7.5) and the fact that

$$\int_s^t a(r)\,\mathrm{d}r = A(t) - A(s)\,,$$

to express formula (7.4) as

$$y = e^{-A(t)} y_I + \int_{t_I}^t e^{-A(t)+A(s)} f(s)\,\mathrm{d}s = e^{-A(t)} y_I + e^{-A(t)} B(t)\,. \tag{7.7}$$

Because $A(t)$ and $B(t)$ are continuously differentiable over (t_L, t_R) with $A'(t)$ and $B'(t)$ given by (7.6), we see that y is also continuously differentiable over (t_L, t_R) with

$$\begin{aligned} y' &= -A'(t)e^{-A(t)} y_I - A'(t)e^{-A(t)} B(t) + e^{-A(t)} B'(t) \\ &= -a(t)e^{-A(t)} y_I - a(t)e^{-A(t)} B(t) + f(t) = -a(t)y + f(t)\,. \end{aligned}$$

Hence, y given by formula (7.4) is a solution of the differential equation in (7.3) over (t_L, t_R). Moreover, because it is clear from (7.4) that $A(t_I) = B(t_I) = 0$, we see that

$$y(t_I) = e^{-A(t_I)} y_I + e^{-A(t_I)} B(t_I) = e^{-0} y_I + e^{-0} \cdot 0 = y_I\,.$$

Hence, y given by formula (7.4) also satisfies the initial condition in (7.3). Therefore y given by (7.4) solves the initial-value problem (7.3). This insures the existence of at least one continuously differentiable solution of initial-value problem (7.3) over (t_L, t_R).

What remains is to prove that this is a unique solution. Let $y(t)$ be any solution of the initial-value (7.3) over (t_L, t_R). We want to show that this solution is the one given by formula (7.4). We can do this by retracing the steps by which formula (7.4) was derived. Because $A(t)$ and $y(t)$ are differentiable over (t_L, t_R) with $A'(t) = a(t)$ and $y'(t) + a(t)y(t) = f(t)$ respecively, we know that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{A(t)} y(t)\right) = e^{A(t)} y'(t) + A'(t)e^{A(t)} y(t) = e^{A(t)}\left(y'(t) + a(t)y(t)\right) = e^{A(t)} f(t)\,.$$

Because $B(t)$ is differentiable over (t_L, t_R) with $B'(t) = e^{A(t)} f(t)$, the above equation can be expressed as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{A(t)} y(t) - B(t)\right) = 0\,.$$

But any differentiable function whose derivative is zero over an interval must be constant over that interval. Therefore there exists a constant c such that

$$e^{A(t)} y(t) - B(t) = c \qquad \text{over } (t_L, t_R)\,.$$

Because $t_I \in (t_L, t_R)$, we may find c by evaluating the above expression at $t = t_I$ and using the fact that $A(t_I) = B(t_I) = 0$ and $y(t_I) = y_I$. We obtain

$$c = e^{A(t_I)} y(t_I) - B(t_I) = e^0 y_I - 0 = y_I \,.$$

Therefore

$$y(t) = e^{-A(t)} \big(y_I + B(t) \big) \,,$$

which is the solution given by formula (7.4). Therefore that solution is the unique solution of initial-value problem (7.3) over (t_L, t_R). □

Remark. Theorem 7.1 proves only the existence and uniqueness of solutions to the initial-value problem (7.3). However, because it gives an explicit formula for the solution, it goes a long way towards establishing the continuous dependence of the solution on the parameters of the problem — namely, on t_I, y_I, $a(t)$, and $f(t)$. We will not formulate such a result here.

7.2.2. *Intervals of Definition.* For linear equations we can usually identify the interval of definition for the solution of the initial-value problem (7.3) by simply looking at $a(t)$ and $f(t)$. Specifically, if $Y(t)$ is the solution of the initial-value problem (7.3) then its interval of definition will be (t_L, t_R) whenever:

- the coefficient $a(t)$ and forcing $f(t)$ are continuous over (t_L, t_R),
- the initial time t_I is in (t_L, t_R),
- either the coefficient $a(t)$ or the forcing $f(t)$ is not defined at both $t = t_L$ and $t = t_R$.

This is because the first two bullets along with the formula (7.4) imply that the interval of definition will be at least (t_L, t_R), while the last two bullets along with Definition 7.1 of solution imply that the interval of definition can be no bigger than (t_L, t_R) because the equation is not defined at both $t = t_L$ and $t = t_R$. This argument can be applied when either $t_L = -\infty$ or $t_R = \infty$.

Example. Give the interval of definition for the solution of the initial-value problem

$$z' + \cot(t)\, z = \frac{1}{\log(t^2)} \,, \qquad z(4) = 3 \,.$$

Solution. The coefficient $\cot(t)$ is not defined at $t = n\pi$ where n is any integer, and is continuous everywhere else. The forcing $1/\log(t^2)$ is not defined at $t = 0$ and $t = 1$, and is continuous everywhere else. Therefore the interval of definition is $(\pi, 2\pi)$ because: both $\cot(t)$ and $1/\log(t^2)$ are continuous over this interval; the initial time is $t = 4$, which is in this interval; $\cot(t)$ is not defined at $t = \pi$ and $t = 2\pi$.

Example. Give the interval of definition for the solution of the initial-value problem

$$z' + \cot(t)\, z = \frac{1}{\log(t^2)} \,, \qquad z(2) = 3 \,.$$

Solution. The interval of definition is $(1, \pi)$ because: both $\cot(t)$ and $1/\log(t^2)$ are continuous over this interval; the initial time is $t = 2$, which is in this interval; $\cot(t)$ is not defined at $t = \pi$ while $1/\log(t^2)$ is not defined at $t = 1$.

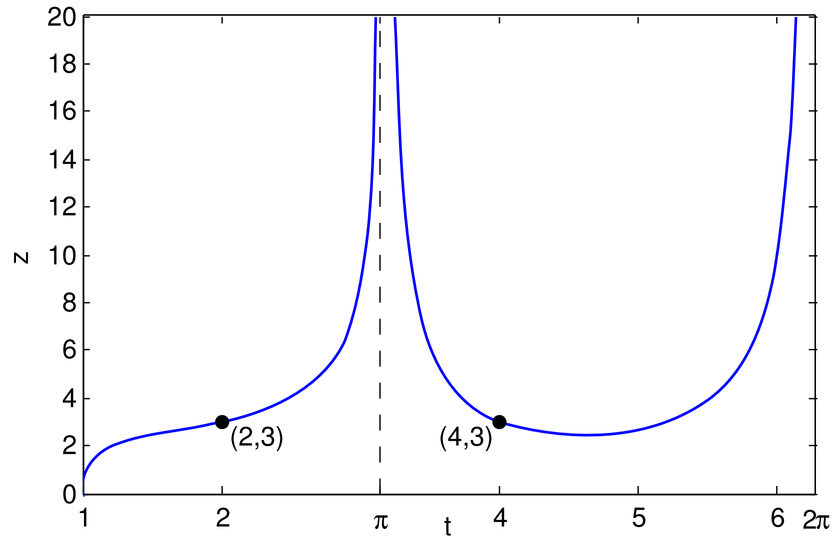

FIGURE 7.1. Solutions of $z' + \cot(t)z = 1/\log(t^2)$ that satisfy the initial conditions $z(2) = 3$ and $z(4) = 3$. The interval of definitions are $(1, \pi)$ and $(\pi, 2\pi)$ respectively. Notice that $z(t)$ blows up as t approaches π and 2π, while $z(t)$ does not blow up but $z'(t)$ does blow up as t approaches 1.

Remark. If $y = Y(t)$ is a solution of (7.3) whose interval of definition is (t_L, t_R) then this does not mean that $Y(t)$ will become undefined at either $t = t_L$ or $t = t_R$ when those endpoints are finite. For example, $y = t^4$ solves the initial-value problem

$$t\,y' - 4y = 0\,, \qquad y(1) = 1\,,$$

and is defined for every t. However, the interval of definition is just $(0, \infty)$ because the initial time is $t = 1$ and normal form of the equation is

$$y' - \frac{4}{t}\,y = 0\,,$$

the coefficient of which is undefined at $t = 0$.

Remark. It is natural to ask why we do not extend our definition of solutions so that $y = t^4$ is considered a solution of the initial-value problem in the preceding remark for every t. For example, we might say that $y = Y(t)$ is a solution provided it is differentiable and satisfies the above equation rather than its normal form. However by this definition the function

$$Y(t) = \begin{cases} t^4 & \text{for } t \geq 0 \\ ct^4 & \text{for } t < 0 \end{cases}$$

also solves the initial-value problem for any c. This shows that because the equation breaks down at $t = 0$, there are many ways to extend the solution $y = t^4$ to $t < 0$. We avoid such complications by requiring the normal form of the equation to be defined.

7.3. **Separable Equations.** We next specialize to *separable* differential equations. The normal form for these initial-value problems is

$$y' = f(t)\, g(y)\,, \qquad y(t_I) = y_I\,. \tag{7.8}$$

These problems are already in the normal form (7.1). They are among the simplest of intial-value problems, but they are more complicated than linear intial-value problems.

7.3.1. *Recipe for Solutions.* Separable differential equations to not have a general formula for explicit solutions the way linear equations do. Rather, they have a recipe that generally leads to implicit solutions. Here we apply that recipe to the initial-value problem (7.8).

If $g(y_I) = 0$ then it is clear that $y(t) = y_I$ is a solution of (7.8) that is defined for every t. Because this solution does not depend on t it is called a *stationary solution*. Every zero of g is also called a *stationary point* because it yields such a stationary solution. These points are also sometimes called either *equilibrium points*, *critical points*, or *fixed points*. This wealth of names reflects the importance of stationary points in the study of differential equations.

If $g(y_I) \neq 0$ then a *nonstationary solution* can be constructed by first putting the differential equation into its so-called *separated differential form*

$$\frac{1}{g(y)}\,\mathrm{d}y = f(t)\,\mathrm{d}t\,.$$

Then integrate both sides of this equation to obtain

$$\int \frac{1}{g(y)}\,\mathrm{d}y = \int f(t)\,\mathrm{d}t\,.$$

This is equivalent to

$$F(t) = G(y) + c\,, \tag{7.9}$$

where F and G satisfy

$$F'(t) = f(t)\,, \quad G'(y) = \frac{1}{g(y)}\,, \quad \text{and } c \text{ is any constant}\,.$$

We then find c by evaluating (7.9) at the initial data (t_I, y_I). This yields $c = F(t_I) - G(y_I)$, whereby (7.9) becomes

$$G(y) - G(y_I) = F(t) - F(t_I)\,. \tag{7.10}$$

Equation (7.10) is called an *implicit* solution of (7.8).

In order to find an explicit solution, we must solve equation (7.10) for y as a function of t. There may be more than one solution of this equation. If so, we must be sure to take the one that satisfies the initial condition. This means we have to find the inverse function of G that recovers y_I — namely a function G^{-1} with the property that

$$G^{-1}\big(G(y)\big) = y \quad \text{for every } y \text{ in an interval within the domain of } G \text{ that contains } y_I\,.$$

In particular, G^{-1} must satisfy $G^{-1}\big(G(y_I)\big) = y_I$. There is a unique inverse function with this property. The solution of the initial-value problem (7.8) is then given by

$$y = G^{-1}\big(G(y_I) + F(t) - F(t_I)\big)\,.$$

This will be valid for all times t in some open interval that contains the initial time t_I. The largest such interval is the interval of definition for the solution.

7.3.2. *Nonuniqueness of Solutions.* Up until now we have mentioned that we must be careful to check that the nonstationary solutions obtained from recipe (7.10) do not hit any of the stationary points, but we have not said why this leads to trouble. The next example illustrates the difficulty that arises.

Example. Find all solutions to the initial-value problem

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 3y^{\frac{2}{3}}\,, \qquad y(0) = 0\,.$$

Solution. We see that $y = 0$ is a stationary point of this equation. Therefore $y(t) = 0$ is a stationary solution whose interval of definition is $(-\infty, \infty)$. However, let us carry out recipe (7.10 for nonstationary solutions to see where it leads. These solutions are given implicitly by

$$t = \int \frac{\mathrm{d}y}{3y^{\frac{2}{3}}} = \int \tfrac{1}{3}y^{-\frac{2}{3}}\,\mathrm{d}y = y^{\frac{1}{3}} + c\,.$$

Upon solving this for y we find $y = (t-c)^3$ where c is an arbitrary constant. Notice that each of these solutions hits the stationary point when $t = c$. The initial condition then implies that $0 = (0-c)^3 = -c^3$, whereby $c = 0$. We thereby have found two solutions of the initial-value problem: $y(t) = 0$ and $y(t) = t^3$!

In fact, as we will now show, there are many more solutions of the initial-value problem! Let a and b be any two numbers such that $a \leq 0 \leq b$ and define $y(t)$ by

$$y(t) = \begin{cases} (t-a)^3 & \text{for } t < a\,, \\ 0 & \text{for } a \leq t \leq b\,, \\ (t-b)^3 & \text{for } b < t\,. \end{cases}$$

We can understand such functions better by looking at their graph in Figure 3.7 below. It is clearly a differentiable function with

$$\frac{\mathrm{d}y}{\mathrm{d}t}(t) = \begin{cases} 3(t-a)^2 & \text{for } t < a\,, \\ 0 & \text{for } a \leq t \leq b\,, \\ 3(t-b)^2 & \text{for } b < t\,, \end{cases}$$

whereby it clearly satisfies the initial-value problem. Its interval of definition is $(-\infty, \infty)$. When $a = b = 0$ this reduces to $y(t) = t^3$.

Similarly, for every $a \leq 0$ we can construct the solution

$$y(t) = \begin{cases} (t-a)^3 & \text{for } t < a\,, \\ 0 & \text{for } a \leq t\,, \end{cases}$$

while for every $b \geq 0$ we can construct the solution

$$y(t) = \begin{cases} 0 & \text{for } t \leq b\,, \\ (t-b)^3 & \text{for } b < t\,. \end{cases}$$

The interval of definition for each of these solutions is also $(-\infty, \infty)$.

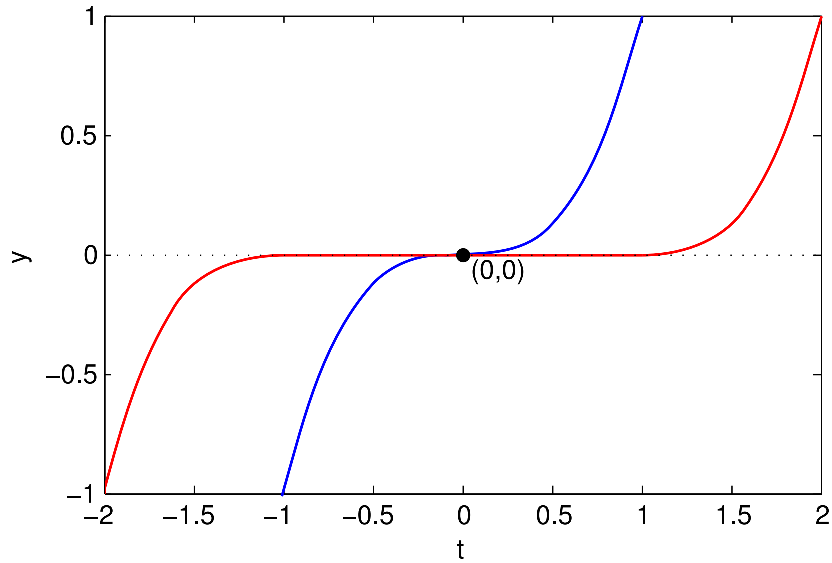

FIGURE 7.2. Illustration of nonuniqueness of solutions for the initial-value problem $y' = 3y^{2/3}$, $y(0) = 0$. The solution $y = t^3$ is shown in blue while the solution given above with $a = -1$ and $b = 1$ is shown in red.

Remark. The above example shows a very important difference between nonlinear and linear equations. Specifically, it shows that for nonlinear equations an initial-value problem may not have a unique solution.

Remark. The above example also illustrates the general danger of simply constructing a solution by a recipe without having any idea whether or not the constructed solution is unique.

7.3.3. *Existence and Uniqueness Theorem.* The nonuniqueness seen in the foregoing example arises because $g(y) = 3y^{\frac{2}{3}}$ does not behave nicely at the stationary point $y = 0$. It is clear that g is continuous at 0, but because $g'(y) = 2y^{-\frac{1}{3}}$, we see that g is not differentiable at 0.

The following fact states the differentiablity of g is enough to ensure that the solution of the initial-value problem exists and is unique. Indeed, the continuity of g plus the differentiablity of g at each stationary point is enough.

Theorem 7.2. *Let $f(t)$ and $g(y)$ be functions defined over the open intervals (t_L, t_R) and (y_L, y_R) respecively such that*

- *f is continuous over (t_L, t_R),*

- *g is continuous over (y_L, y_R),*
- *g is differentiable at each of its zeros in (y_L, y_R).*

Then for every inital time t_I in (t_L, t_R), and every initial value y_I in (y_L, y_R) there exists a unique solution $y = Y(t)$ to the initial-value problem (7.8) that is defined over every time interval (a, b) such that

- *t_I is in (a, b),*
- *(a, b) is contained within (t_L, t_R),*
- *$Y(t)$ remains within (y_L, y_R) while t is in (a, b).*

Moreover, this solution is continuously differentiable and is determined by our recipe. This means either $g(y_I) = 0$ and $Y(t) = y_I$ is a stationary solution, or $g(y_I) \neq 0$ and $Y(t)$ is a nonstationary solution that satisfies

$$G\big(Y(t)\big) = F(t)\,, \qquad Y(t_I) = y_I\,,$$

where $F(t)$ is defined for every t in (t_L, t_R) by the definite integral

$$F(t) = \int_{t_I}^{t} f(s)\,\mathrm{d}s\,,$$

while $G(y)$ is defined by the definite integral

$$G(y) = \int_{y_I}^{y} \frac{1}{g(x)}\,\mathrm{d}x\,,$$

whenever the point y is in (y_L, y_R) and neither y nor any point between y and y_I is a stationary point of g.

In particular, if f is continuous over $(-\infty, \infty)$ while g is differentiable over $(-\infty, \infty)$ then the initial-value problem (7.8) has a unique solution that either exists for all time or "blows up" at a finite time. Moreover, this solution is continuously differentiable and is determined by our recipe. This "blow up" behavior can be seen when $g(y) = y^2$ or $g(y) = 1 + y^2$. Indeed, it can be seen whenever $g(y)$ is a polynomial of degree two or more.

Remark. The above theorem implies that if the initial point y_I lies between two stationary points within (y_L, y_R) then the solution $Y(t)$ exists for all t in (t_L, t_R). This is because the uniqueness assertion implies $Y(t)$ cannot cross any stationary point, and therefore is trapped within (y_L, y_R). In particular, if g is differentiable over $(-\infty, \infty)$ then the only solutions that might "blows up" in a finite time are those that are not trapped above and below by stationary points.

Example. For $y' = y^2$ the only stationary point is $y = 0$. Because $g(y) = y^2 > 0$ when $y \neq 0$ we see that every nonstationary solution $Y(t)$ will be an increasing function of t. This fact is verified by the explicit solution for the initial condition $y(0) = y_I$, which is

$$Y(t) = \frac{y_I}{1 - y_I t}\,.$$

When $y_I > 0$ the interval of definition is $(-\infty, 1/y_I)$ and we see that $Y(t) \to +\infty$ as $t \to 1/y_I$ while $Y(t) \to 0$ as $t \to -\infty$. In this case the solution is trapped below as $t \to -\infty$ by the stationary point $y = 0$. Similarly, when $y_I < 0$ the interval of definition is $(1/y_I, \infty)$ and we see that $Y(t) \to -\infty$ as $t \to 1/y_I$ while $Y(t) \to 0$ as $t \to \infty$. In this case the solution is trapped above as $t \to \infty$ by the stationary point $y = 0$. Figure 3.2 shows these solutions for the initial values $y_I = -2$ and $y_I = 2$.

Example. For $y' = 4y - y^3$ the stationary points are $y = -2$, $y = 0$, and $y = 2$. For the initial condition $y(0) = y_I$ we obtain

$$c = -\frac{1}{8}\log\left(\frac{y_I^2}{|4 - y_I^2|}\right) .$$

Typical nonstationary solution are plotted in Figure 3.1. That figure shows that each nonstationary solution $Y(t)$ is trapped within one of the intervals $(-\infty, -2)$, $(-2, 0)$, $(0, 2)$, or $(2, \infty)$. Notice that for the solutions trapped within either $(-\infty, -2)$ or $(2, \infty)$ we have $y_I^2 > 4$, whereby $c < 0$.

- If $Y(t)$ lies within $(-\infty, -2)$ then its interval of definition is (c, ∞). Moreover, $Y(t)$ is increasing with

$$\lim_{t\to c} Y(t) = -\infty , \qquad \lim_{t\to\infty} Y(t) = -2 .$$

- If $Y(t)$ lies within $(-2, 0)$ then its interval of definition is $(-\infty, \infty)$. Moreover, $Y(t)$ is decreasing with

$$\lim_{t\to-\infty} Y(t) = 0 , \qquad \lim_{t\to\infty} Y(t) = -2 .$$

- If $Y(t)$ lies within $(0, 2)$ then its interval of definition is $(-\infty, \infty)$. Moreover, $Y(t)$ is increasing with

$$\lim_{t\to-\infty} Y(t) = 0 , \qquad \lim_{t\to\infty} Y(t) = 2 .$$

- If $Y(t)$ lies within $(2, \infty,)$ then its interval of definition is (c, ∞). Moreover, $Y(t)$ is decreasing with

$$\lim_{t\to c} Y(t) = \infty , \qquad \lim_{t\to\infty} Y(t) = 2 .$$

Remark. Even in such cases where we cannot find an explicit inverse function of $G(y)$ we often can determine the interval of definition of the solution directly from the equation

$$G(y) = F(t) , \quad \text{where} \quad F(t) = \int_{t_I}^{t} f(s)\, \mathrm{d}s , \quad G(y) = \int_{y_I}^{y} \frac{1}{g(x)}\, \mathrm{d}x .$$

For example, if $g(y) > 0$ over $[y_I, \infty)$ then $G(y)$ will be increasing over $[y_I, \infty)$ and the solution $Y(t)$ will be defined over the largest interval (a, b) such that t_I is in (a, b), (a, b) is contained within (t_L, t_R), and

$$F(t) < \lim_{y\to+\infty} G(y) .$$

If the above limit is finite and equal to $F(b)$ then the solution “blows up” as $t \to b^-$.

Similarly, if $g(y) > 0$ over $(-\infty, y_I]$ then $G(y)$ will be increasing over $(-\infty, y_I]$ and the solution $Y(t)$ will be defined over the largest interval (a, b) such that t_I is in (a, b), (a, b) is contained within (t_L, t_R), and

$$\lim_{y \to -\infty} G(y) < F(t) .$$

If the above limit is finite and equal to $F(b)$ then the solution "blows down" as $t \to b^-$.

7.4. **General Equations.** In this chapter we consider initial-value problems for first-order differential equations that are in the normal form

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t, y) , \qquad y(t_I) = y_I . \tag{7.11}$$

We will give conditions on $f(t, y)$ that ensure this problem has a unique solution. This theory is used to develop methods by which we can study solutions of these problems when analytical methods either do not apply or become complicated.

7.4.1. *Picard Existence and Uniqueness Theorem.* For the separable initial-value problem (7.8) we added the hypothesis that $g(y)$ be differentiable at every stationary point in order to insure that its solutions were unique. The question now is what hypothesis can we add to insure the more general initial-value problem (7.11) has unique solutions? In order to state our hypotheses we will give three definitions.

We begin with the notion of continuity.

Definition 7.2. *Let S be a set in the ty-plane. A function $f(t, y)$ defined over S is said to be* continuous *over S if for every point $(t, y) \in S$ and every sequence $\{(t_n, y_n)\} \subset S$ such that $t_n \to t$ and $y_n \to y$ we have $f(t_n, y_n) \to f(t, y)$.*

Next, we define the interior of a set, which a picture should help clarify.

Definition 7.3. *Let S be a set in the ty-plane. A point (t_o, y_o) is said to be in the* interior *of S if there exists a rectangle $(t_L, t_R) \times (y_L, y_R)$ that contains the point (t_o, y_o) and also lies within the set S.*

Finally, we define what it means for a function to be locally Lipschitz in y.

Definition 7.4. *Let S be a set in the ty-plane. Let $f(t, y)$ be a function defined over S. We say that $f(t, y)$ is* locally Lipschitz *in y if for every bounded rectangle $B = [t_L, t_R] \times [y_L, y_R]$ that lies within the interior of S there exists a constant L such that*

$$|f(t, y) - f(t, z)| \le L|y - z| \quad \textit{for every } (t, y) \textit{ and } (t, y) \textit{ in } B .$$

We call any such L a Lipschitz constant in y *for B.*

Remark. If $f(t, y)$ is differentiable in y over a rectangle $B = [t_L, t_R] \times [y_L, y_R]$ and $\partial_y f(t, y)$ is bounded over B then the smallest Lipschitz constant in y for B is given by

$$L = \sup\big\{ |\partial_y f(t, y)| \, : \, (t, y) \in B \big\} .$$

This fact is often proved in beginning analysis courses. In practice, $f(t, y)$ is often continuously differentiable over B, in which case $\partial_y f(t, y)$ exists and is bounded over B

We are now ready to state the Picard existence and uniqueness theorem. Its proof will be given over the remainder of this section.

Theorem 7.3. *Let $f(t,y)$ be a function defined over a set S in the ty-plane such that*
- *f is continuous over S,*
- *f is locally Lipschitz in y over S.*

Then for every inital time t_I and every initial value y_I such that (t_I, y_I) is in the interior of S there exists a unique solution $y = Y(t)$ to initial-value problem (7.11) that is defined over some time interval (a,b) such that
- *t_I is in (a,b),*
- *$\{(t, Y(t)) : t \in (a,b)\}$ lies within the interior of S.*

Moreover, $Y(t)$ extends to the largest such interval and $Y'(t)$ is continuous over that interval.

Remark. This is not the most general theorem we could state, but it is one that applies to many equations you will face and is easy to apply. It asserts that $Y(t)$ will exist until $(t, Y(t))$ leaves S.

As the following examples show, applying this theorem is often a matter of simply checking that $f(t,y)$ and $\partial_y f(t,y)$ are continuous over the set S where they are defined. When this is the case, Theorem 7.3 ensures that for every initial data (t_I, y_I) in the interior of S the initial-value problem has a unique solution $y = Y(t)$ that is defined over some time interval (a,b) that contains t_I.

Example. Determine (t_I, y_I) for which a unique solution exists to the initial-value problem

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\log(|y|)}{1 + t^2 - y^2}\,, \qquad y(t_I) = y_I\,.$$

Solution. Because $f(t,y) = \frac{\log(|y|)}{1+t^2-y^2}$ is defined everywhere except where $1 + t^2 - y^2 = 0$ or $y = 0$, we try taking S to be all points in the ty-plane except those points on the hyperbola $1 + t^2 - y^2 = 0$ (where $y = \pm\sqrt{1+t^2}$) and those points on the t-axis (where $y = 0$). Clearly, f is continuous over S, f is differentiable with respect to y over S with

$$\partial_y f(t,y) = \frac{1}{1 + t^2 - y^2}\,\frac{1}{y} + \frac{\log(|y|)}{(1 + t^2 - y^2)^2}\,2y\,,$$

and $\partial_y f$ is continuous over S. Every point in S is also in the interior of S. Therefore Theorem 7.3 insures that for every (t_I, y_I) in S the initial-value problem has a unique solution $y = Y(t)$ that is defined over some time interval (a,b) that contains t_I. The solution extends to the largest interval (a,b) for which $(t, Y(t))$ remains within S.

Example. Determine (t_I, y_I) for which a unique solution exists to the initial-value problem

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \sqrt{1 + t^2 + y^2}\,, \qquad y(t_I) = y_I\,.$$

Solution. Because $f(t,y) = \sqrt{1 + t^2 + y^2}$ is defined over $(-\infty, \infty) \times (-\infty, \infty)$, we try taking S to be the entire ty-plane. Clearly, f is continuous over S, f is differentiable with respect to y over S with

$$\partial_y f(t,y) = \frac{y}{\sqrt{1+t^2+y^2}}\,,$$

and $\partial_y f$ is continuous over S. Every point in S is also in the interior of S. Therefore Theorem 7.3 insures that for every (t_I, y_I) in the ty-plane the initial-value problem has a unique solution $y = Y(t)$ that is defined over some time interval (a, b) that contains t_I. Either the solution extends to all t or $Y(t)$ blows up in finite time because those are the only ways for $(t, Y(t))$ to leave S. (In fact, it extends to all t.)

Remark. The initial-value problems in the examples above cannot be solved by analytic methods. However, Theorem 7.3 insures that their solutions exist and are unique.

7.4.2. *Integral Formulation.* The theory of solutions for linear and separable equations rested heavily upon the fact that we had formulas that gave either explicit or implicit solutions to those equations. There are no such formulas in the general setting we are now considering. The role that those formulas played in those theories will now be played by the following *integral formulation* of the intial-value problem (7.11).

Let us suppose that $y(t)$ is a classical solution of initial-value problem (7.11) over a time interval (t_L, t_R) that contains t_I. Moreover, let us suppose that $f(t, y(t))$ is continuous over (t_L, t_R). Because $y'(t) = f(t, y(t))$, this means we are assuming that $y(t)$ is continuously differentiable over (t_L, t_R). Then for every $t \in (t_L, t_R)$ we may integrate the differential equation between t_I and t to see that

$$y(t) = y_I + \int_{t_I}^{t} f(s, y(s))\,\mathrm{d}s\,. \tag{7.12}$$

Therefore every classical solution of initial-value problem (7.11) that satisfies the foregoing assumptions is also a solution of integral equation (7.12).

The beauty of the above observation is that under mild assumptions on $f(t, y)$, being a solution of integral equation (7.12) is equivalent to being a solution of initial-value problem (7.11).

Lemma 7.1. *Let the function $f(t, y)$ be continuous over a set S in the ty-plane. Let (t_I, y_I) be in the interior of S. Let (a, b) be a time interval and $Y(t)$ be continuous over (a, b) such that*

- *t_I is in (a, b),*
- *$\{(t, Y(t)) : t \in (a, b)\}$ lies within the interior of S.*

Then $y = Y(t)$ is a classical solution of initial-value problem (7.11) over (a, b) if and only if it is a solution of integral equation (7.12). When this is the case $Y(t)$ is continuously differentiable over (a, b).

Proof. Let $y = Y(t)$ be a classical solution of initial-value problem (7.11) over (a, b). Because $Y(t)$ is continuous over (a, b), $\{(t, Y(t)) : t \in (a, b)\}$ lies within the interior of S, and $f(t, y)$ is continuous over S, we know that $f(t, y(t))$ is continuous over (a, b). Then the argument given above (7.12) shows that $y = Y(t)$ is a solution of the integral equation (7.12) over (a, b) that is continuously differentiable.

Conversely, let $y = Y(t)$ be a solution of integral equation (7.12) over (a, b). Because $Y(t)$ is continuous over (a, b), $\{(t, Y(t)) : t \in (a, b)\}$ lies within the interior of S, and $f(t, y)$ is continuous over S, we know that $f(t, y(t))$ is continuous over (a, b). The Second Fundamental Theorem of Calculus then implies that the right-hand side of (7.12) is continuously differentiable over (a, b), which implies that $Y(t)$ is continuously differentiable over (a, b). Moreover, for every $t \in (a, b)$ we have

$$Y'(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(y_I + \int_{t_I}^{t} f(s, Y(s))\,\mathrm{d}s \right) = f(t, Y(t))\,.$$

It is also clear from (7.12) that $Y(t_I) = y_I$. Therefore $y = Y(t)$ is a classical solution of initial-value problem (7.11) over (a, b) that is continuously differentiable. □

Remark. The idea behind this lemma is that it will be easier to prove that integral equation (7.12) has a solution than to directly prove that initial-value problem (7.11) has a solution because we only have to work with continuous functions rather than differentiable functions.

7.4.3. *Gronwall Lemma and Uniqueness.* First we will illustrate the role played by the locally Lipschitz hypothesis in proving the uniqueness asserted by the Picard Theorem.

Let $y = Y(t)$ and $y = Z(t)$ be continous solutions of the integral equation (7.12) over rectangle $B = [t_I, t_I + h] \times [y_L, y_R]$ for some $h > 0$. Let L be a Lipschitz constant in y for B. Then we have

$$|Y(t) - Z(t)| \leq \int_{t_I}^{t} |f(s, Y(s)) - f(s, Z(s))|\,\mathrm{d}s \leq L \int_{t_I}^{t} |Y(s) - Z(s)|\,\mathrm{d}s\,. \tag{7.13}$$

We want to show that this implies $|Y(t) - Z(t)| = 0$ over $[t_I, t_I + h]$. That would imply that $Y(t) = Z(t)$ over $[t_I, t_I + h]$, thereby proving the uniqueness of the solution.

The way we will show that $|Y(t) - Z(t)| = 0$ over $[t_I, t_I + h]$ is by using the Gronwall Lemma, which we now state and prove.

Lemma 7.2. (Gronwall Lemma.) *Let $\alpha(t)$, $\beta(t)$, and $\eta(t)$ be nonnegative, continuous functions over $[t_I, t_I + h]$ that satisfy the integral inequality*

$$\eta(t) \leq \alpha(t) + \alpha(t) \int_{t_I}^{t} \beta(s)\eta(s)\,\mathrm{d}s\,. \tag{7.14}$$

Then $\eta(t)$ is bounded for every $t \in [t_I, t_I + h]$ by

$$\eta(t) \leq \alpha(t) \exp\left(\int_{t_I}^{t} \alpha(s)\beta(s)\mathrm{d}s \right)\,. \tag{7.15}$$

Proof. Define $\phi(t)$ for every $t \in [t_I, t_I + h]$ by

$$\phi(t) = 1 + \int_{t_I}^{t} \beta(s)\eta(s)\,\mathrm{d}s\,.$$

Because $\beta(t)$ and $\eta(t)$ are continuous over $[t_I, t_I + h]$, the Second Fundamental Theorem of Calculus implies that $\phi(t)$ is continuously differentiable over $[t_I, t_I + h]$ with $\phi'(t) =$

$\beta(t)\eta(t)$. Then because $\beta(t)$ is nonnegative and because the integral inequality (7.14) can be recast as $\eta(t) \leq \alpha(t)\phi(t)$, we obtain

$$\phi'(t) = \beta(t)\eta(t) \leq \alpha(t)\beta(t)\phi(t)\,.$$

But this is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\exp\left(-\int_{t_I}^{t} \alpha(s)\beta(s)\mathrm{d}s\right)\phi(t)\right) \leq 0\,.$$

Because $\phi(t_I) = 1$, the above inequality implies that

$$\phi(t) \leq \exp\left(\int_{t_I}^{t} \alpha(s)\beta(s)\mathrm{d}s\right)\,.$$

When this bound is placed into $\eta(t) \leq \alpha(t)\phi(t)$, we obtain bound (7.15). □

We now let $\delta > 0$ be arbitrary and apply the Gronwall Lemma with $\alpha(t) = \delta$, $\beta(t) = L/\delta$, and $\eta(t) = |Y(t) - Z(t)|$. Then integral inequality (7.14) is satisfied by (7.13). The Gronwall Lemma then gives bound (7.15), which is

$$|Y(t) - Z(t)| \leq \delta\, e^{L(t-t_I)} \quad \text{for every } t \in [t_I, t_I + h]\,.$$

But $\delta > 0$ was arbitrary. Therefore $|Y(t) - Z(t)| = 0$, completing our uniqueness argument.

7.4.4. *Picard Iteration.* It is now time to show that integral equation (7.12) has a continuous solution $Y(t)$ over a time interval $[t_I, t_I + h]$ for a sufficiently small $h > 0$. This will be done by constructing a sequence $\{Y_n(t)\}$ of approximate solutions and showing that they converge to a solution of integral equation (7.12). The approximate solutions that we will use are defined iteratively as follows.

Let $Y_0(t)$ be the first approximate solution. A common choice is $Y_0(t) = y_I$ because it satisfies the initial condition and is extremely simple, however any choice that satisfies the initial condition will work. Given $Y_n(t)$ we construct $Y_{n+1}(t)$ by

$$Y_{n+1}(t) = y_I + \int_{t_I}^{t} f(s, Y_n(s))\,\mathrm{d}s \quad \text{for every } t \in [t_I, t_I + h]\,. \tag{7.16}$$

In other words, we construct $Y_{n+1}(t)$ to be the right-hand side of integral equation (7.12) evaluated at $Y_n(t)$. This method of constructing a sequence of approximate solutions is commonly called *Picard iteration* or the *Picard method of successive approximations*. The approximations $Y_n(t)$ are commonly called *Picard iterates* or *Picard approximations*.

Before we can hope to show that the Picard iterates converge to a solution of integral equation (7.12), we must show that they exist! The problem is that we have to make sure that each $Y_n(t)$ takes values such that $(t, Y_n(t))$ remains within the interior of S for every $t \in [t_I, t_I + h]$. We will do this by picking $h > 0$ sufficiently small.

Lemma 7.3. *Let S and $f(t, y)$ be as in the Picard Theorem. Then there exists $h > 0$ such that each Picard iterate $Y_n(t)$ is continuous over $[t_I, t_I + h]$ and takes values such that $(t, Y_n(t))$ remains within the interior of S for every $t \in [t_I, t_I + h]$.*

Proof. Because the point (t_I, y_I) lies in the interior of S there exists a rectangle $(t_L, t_R) \times (y_L, y_R)$ that contains the point (t_I, y_I) and is contained in the set S. We then pick $\delta > 0$ and $\eta > 0$ such that

$$R = [t_I, t_I + \eta] \times [y_I - \delta, y_I + \delta] \subset (t_L, t_R) \times (y_L, y_R)\,. \tag{7.17}$$

Because $f(t, y)$ is continuous over the closed, bounded set R, it takes on its extreme values. Let

$$M = \max\big\{|f(t, y)| \,:\, (t, y) \in R\big\}\,. \tag{7.18}$$

Now pick $h \in (0, \eta]$ so that $Mh \leq \delta$. We claim that every Picard iterate is continuous over $[t_I, t_I + h]$ and satisfies the bound

$$|Y_n(t) - y_I| \leq \delta \quad \text{for every } t \in [t_I, t_I + h]\,. \tag{7.19}$$

We prove this by induction. By choosing $Y_0(t) = y_I$, this obviously holds for $n = 0$. Now suppose that it holds for $Y_n(t)$. We want to show that it holds for $Y_{n+1}(t)$. Because it holds for $Y_n(t)$, we know that $(t, Y_n(t)) \in R$ for every $t \in [t_I, t_I + h]$, which by (7.18) implies that $|f(t, Y_n(t))| \leq M$ for every $t \in [t_I, t_I + h]$. But then from definition (7.16) of $Y_{n+1}(t)$ we see that

$$|Y_{n+1}(t) - y_I| \leq \int_{t_I}^{t} |f(s, Y_n(s))|\, \mathrm{d}s \leq M(t - t_I) \leq Mh \leq \delta\,.$$

Therefore bound (7.19) holds for $Y_{n+1}(t)$. In addition, $Y_{n+1}(t)$ is continous over $[t_I, t_I + h]$. Indeed, because $f(t, Y_n(t))$ is continuous over $[t_I, t_I + h]$, it is clear from definition (7.16) that $Y_{n+1}(t)$ is continously differentiable over $[t_I, t_I + h]$. By induction, each Picard iterate is continuous over $[t_I, t_I + h]$ and satisfies bound (7.19). □

Remark. The restriction of h by the conditions $h < \eta$ and $Mh \leq \delta$ has simple interpretation of keeps $(t, y(t))$ within the rectangle R. Specifically, the condition $h < \eta$ keeps t within $[t_I, t_I + \eta]$, while the condition $Mh \leq \delta$ keeps $y(t)$ within $[y_I - \delta, y_I + \delta]$. Indeed, because M is the maximum absolute value of y', the maximum amount that $y(t)$ can change over a time interval of length h is Mh.

It remains to be shown that the Picard iterates converge to a solution of integral equation (7.12). This has two steps: showing the Picard iterates converge, and showing the limiting function solves integral equation (7.12). The first step uses the *Cauchy Criterion* for the convergence of sequences. The second step uses the fact that the convergence is *uniform*. These are concepts are often introduced in analysis courses.

The first step is contained in the following lemma.

Lemma 7.4. *Let S and $f(t, y)$ be as in the Picard Theorem. Let $h > 0$ and $\delta > 0$ be as in the proof of Lemma 7.3. Let $\{Y_n\}$ be the sequence of Picard iterates constructed in Lemma 7.3. Then for each $t \in [t_I, t_I + h$ the sequence of real numbers $\{Y_n(t)\}$ is Cauchy, and thereby is convergent by the Cauchy Criterion. Moreover, for every $n > m$ and every $t \in [t_I, t_I + h$ we have the bound*

$$|Y_n(t) - Y_m(t)| \leq \frac{(Lh)^m}{m!}\, \delta\, e^{Lh}\,, \tag{7.20}$$

where L is a Lipshitz constant in y for the rectangle $B = [t_I, t_I + h] \times [y_I - \delta, y + \delta]$.

Proof. Let $\epsilon > 0$ be arbitrary. Because the Picard iterates are generated by definition (7.16), for every $n \geq 1$ and $t \in [t_I, t_I + h]$ we have

$$|Y_{n+1}(t) - Y_n(t)| \leq \int_{t_I}^{t} |f(s, Y_n(s)) - f(s, Y_{n-1}(s))| \, \mathrm{d}s \leq L \int_{t_I}^{t} |Y_n(s) - Y_{n-1}(s)| \, \mathrm{d}s \, .$$

Because $|Y_1(t) - y_I| \leq \delta$, we can use this inequality to show by induction that for every $n \geq 0$ and every $t \in [t_I, t_I + h]$ we have

$$|Y_{n+1}(t) - Y_n(t)| \leq \frac{L^n (t - t_I)^n}{n!} \, \delta \, .$$

Then for every $n > m$ and every $t \in [t_I, t_I + h]$ we have

$$\begin{aligned}
|Y_n(t) - Y_m(t)| &\leq \sum_{k=m}^{n-1} |Y_{k+1}(t) - Y_k(t)| \\
&\leq \sum_{k=m}^{n-1} \frac{L^k (t - t_I)^k}{k!} \, \delta \\
&\leq \frac{L^m (t - t_I)^m}{m!} \, \delta \sum_{k=m}^{n-1} \frac{L^{k-m} (t - t_I)^{k-m}}{(k-m)!} \\
&= \frac{L^m (t - t_I)^m}{m!} \, \delta \sum_{k=0}^{n-m-1} \frac{L^k (t - t_I)^k}{k!} \\
&\leq \frac{L^m (t - t_I)^m}{m!} \, \delta \, e^{L(t - t_I)} \leq \frac{(Lh)^m}{m!} \, \delta \, e^{Lh} \, .
\end{aligned}$$

This proves (7.20). Finally, by picking m large enough the right-hand side will be smaller than ϵ. Hence, for each $t \in [t_I, t_I + h]$ the sequence of real numbers $\{Y_n(t)\}$ is Cauchy, and thereby is convergent by the Cauchy Criterion. □

The second step is contained in the following lemma.

Lemma 7.5. *Let S and $f(t, y)$ be as in the Picard Theorem. Let $h > 0$ and $\delta > 0$ be as in the proof of Lemma 7.3. Let $\{Y_n\}$ be the sequence of Picard iterates constructed in Lemma 7.3. Define $Y(t)$ by*

$$Y(t) = \lim_{t \to \infty} Y_n(t) \qquad \textit{for every } t \in [t_I, t_I + h] \, .$$

Then $Y(t)$ is continuous over $[t_I, t_I + h]$ and satisfies integral equation (7.12).

Proof. By Lemma 7.4 the sequence $\{Y_n(t)\}$ is convergent for every $t \in [t_I, t_I + h]$, so that the limit above that defines the function $Y(t)$ exists over this interval. We can then let $n \to \infty$ in (7.20) to obtain the uniform bound

$$|Y(t) - Y_m(t)| \leq \frac{(Lh)^m}{m!} \, \delta \, e^{Lh} \, , \tag{7.21}$$

where L is a Lipshitz constant in y for the rectangle $B = [t_I, t_I + h] \times [y_I - \delta, y + \delta]$. The right-hand side of this bound is independent of t and vanishes as $m \to \infty$. This shows that $Y_n(t)$ converges to $Y(t)$ uniformly in t.

Next we argue that Y is continuous over $[t_I, t_I+h]$. This follows from the general fact that the uniform limit of a sequence of continuous functions is also continuous. This fact is often taught in analysis courses. Rather than use this general fact, we present an independent proof that Y is continuous over $[t_I, t_I+h]$ which only requires that you know the ϵ-δ definition of continuity at a point. The triangle inequality and the uniform bound (7.21) imply that for every $s, t \in [t_I, t_I+h]$

$$\begin{aligned}|Y(s) - Y(t)| &\leq |Y(s) - Y_n(s)| + |Y_n(s) - Y_n(t)| + |Y_n(t) - Y(t)| \\ &\leq \frac{(Lh)^n}{n!}\,\delta\, e^{Lh} + |Y_n(s) - Y_n(t)| + \frac{(Lh)^n}{n!}\,\delta\, e^{Lh}\,.\end{aligned}$$

Let $t \in [t_I, t_I+h]$. Let $\epsilon > 0$. Pick n large enough so that

$$\frac{(Lh)^n}{n!}\,\delta\, e^{Lh} < \frac{\epsilon}{3}\,.$$

Because Y_n is continuous at t we can pick $\delta_\epsilon > 0$ small enough so that for every $s \in [t_I, t_I+h]$

$$|s - t| < \delta_\epsilon \implies |Y_n(s) - Y_n(t)| < \frac{\epsilon}{3}\,.$$

By combining the three lines above we see that

$$|s - t| < \delta_\epsilon \implies |Y(s) - Y(t)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon\,.$$

Hence, Y is continuous at t. But $t \in [t_I, t_I+h]$ was arbitrary. Therefore Y is continuous over $[t_I, t_I+h]$.

Fianlly, we show that $Y(t)$ solves integral equation (7.12) by letting $n \to \infty$ in the defining relation (7.16) of the Picard iterates. It is clear that

$$\lim_{n\to\infty} Y_{n+1}(t) = Y(t)\,.$$

By using the uniform bound we have

$$\begin{aligned}\left|\int_{t_I}^t f(s, Y_n(s))\,\mathrm{d}s - \int_{t_I}^t f(s, Y(s))\,\mathrm{d}s\right| &\leq \int_{t_I}^t |f(s, Y_n(s)) - f(s, Y(s))|\,\mathrm{d}s \\ &\leq L\int_{t_I}^t |Y_n(s) - Y(s)|\,\mathrm{d}s \\ &\leq \frac{(Lh)^{n+1}}{n!}\,\delta\, e^{Lh}\,,\end{aligned}$$

which implies that

$$\lim_{n\to\infty}\int_{t_I}^t f(s, Y_n(s))\,\mathrm{d}s = \int_{t_I}^t f(s, Y(s))\,\mathrm{d}s\,.$$

Therefore $Y(t)$ solves integral equation (7.12). □

Lemmas 7.1 through 7.5 combine to give a proof of the main conclusions of the Picard Theorerm, Theorem 7.3. □