Second In-Class Exam Solutions
Math 246, Professor David Levermore
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(1) [4] Give the interval of definition for the solution of the initial-value problem

$$w''' - \frac{5}{t}\,w' + \frac{\cos(5t)}{2+t}\,w = \frac{e^t}{4-t}\,, \qquad w(2) = w'(2) = w''(2) = 0\,.$$

Solution. The equation is linear and is already in normal form. The coefficient of w' is undefined at $t = 0$ and is continuous elsewhere. The coefficient of w is undefined at $t = -2$ and is continuous elsewhere. The forcing is undefined at $t = 4$ and is continuous elsewhere. Therefore the interval of definition is $(0, 4)$ because:

- the initial time $t = 2$ is in the interval $(0, 4)$;
- all the coefficients and the forcing are continuous over $(0, 4)$;
- the coefficient of w' is undefined at $t = 0$, the left endpoint of $(0, 4)$;
- and the forcing is undefined at $t = 4$, the right endpoint of $(0, 4)$.

(2) [12] Let L be a linear ordinary differential operator with constant coefficients. Suppose that all the roots of its characteristic polynomial (listed with their multiplicities) are $-2 + i5$, $-2 + i5$, $-2 - i5$, $-2 - i5$, $i4$, $-i4$, 3, 3, 3, 0, 0, 0.

(a) [2] Give the order of L.
(b) [10] Give a general real solution of the homogeneous equation $\mathrm{L}v = 0$.

Solution (a). There are 12 roots listed above, so the degree of the characteristic polynomial is 12, whereby the order of L is also 12.

Solution (b). A general solution is

$$v(t) = c_1 e^{-2t}\cos(5t) + c_2 e^{-2t}\sin(5t) + c_3 t\, e^{-2t}\cos(5t) + c_4 t\, e^{-2t}\sin(5t)$$
$$+ c_5\cos(4t) + c_6\sin(4t) + c_7 e^{3t} + c_8 t e^{3t} + c_9 t^2 e^{3t} + c_{10} + c_{11} t + c_{12} t^2\,.$$

Here the fundamental set of solutions is generated as follows:

- the double conjugate pair $-2 \pm i5$ yields
$$e^{-2t}\cos(5t)\,, \quad e^{-2t}\sin(5t)\,, \quad t\, e^{-2t}\cos(5t)\,, \quad \text{and} \quad t\, e^{-2t}\sin(5t)\,;$$
- the single conjugate pair $\pm i4$ yields $\cos(4t)$ and $\sin(4t)$;
- the triple real root 3 yields e^{3t}, $t\, e^{3t}$, and $t^2 e^{3t}$;
- the triple real root 0 yields 1, t, and t^2.

(3) [4] Suppose that $X_1(t)$, $X_2(t)$, and $X_3(t)$ are solutions of the differential equation

$$x''' - 3x'' - \cos(t)x' + e^t x = 0\,,$$

Suppose you know that $W[X_1, X_2, X_3](0) = 5$. What is $W[X_1, X_2, X_3](t)$?

Solution. The Abel Theorem states that $w(t) = W[X_1, X_2, X_3](t)$ satisfies the first-order homogeneous linear equation $w' - 3w = 0$. It follows that $w(t) = w(0)e^{3t}$. Because $w(0) = W[X_1, X_2, X_3](0) = 5$, we obtain $w(t) = 5e^{3t}$. Therefore

$$W[X_1, X_2, X_3](t) = 5e^{3t}\,.$$

(4) [12] The functions $\cos(3t)$ and $\sin(3t)$ are a fundamental set of solutions to $u''+9u=0$.
(a) [9] Find the solution $U(t)$ to the general initial-value problem

$$u'' + 9u = 0\,, \qquad u(0) = u_0\,, \quad u'(0) = u_1\,.$$

(b) [3] Find the associated natural fundamental set of solutions to $u'' + 9u = 0$.

Solution (a). Because we are given that $\cos(3t)$ and $\sin(3t)$ is a fundamental set of solutions to $u'' + 9u = 0$, a general solution is $U(t) = c_1\cos(3t) + c_2\sin(3t)$. Because $U'(t) = -3c_1\sin(3t) + 3c_2\cos(3t)$, the initial conditions imply

$$u_0 = U(0) = c_1\,, \qquad\qquad u_1 = U'(0) = 3c_2\,.$$

We solve these equations to obtain

$$c_1 = u_0\,, \qquad\qquad c_2 = \tfrac{1}{3}u_1\,.$$

Therefore the solution to the general initial-value problem is

$$U(t) = u_0\cos(3t) + u_1\tfrac{1}{3}\sin(3t)\,.$$

Solution (b). We see from the above solution to the general initial-value problem that the associated natural fundamental set of solutions is

$$N_0(t) = \cos(3t)\,, \qquad N_1(t) = \tfrac{1}{3}\sin(3t)\,.$$

(5) [9] Give a general real solution of the equation

$$\mathrm{D}^2 w - 5\mathrm{D}w + 6w = 20\sin(4t)\,, \qquad \text{where} \quad \mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}t}\,.$$

Solution. This is a nonhomogeneous linear equation with constant coefficients. Its characteristic polynomial is

$$p(z) = z^2 - 5z + 6 = (z-2)(z-3)\,,$$

which has the two simple real roots 2 and 3. Therefore a general solution of the associated homogeneous equation is

$$w_H(t) = c_1 e^{2t} + c_2 e^{3t}\,.$$

The forcing $20\sin(4t)$ has characteristic form with degree $d = 0$ and characteristic $\mu + i\nu = i4$, which is a root of $p(z)$ of multiplicity $m = 0$. Therefore we can use either Key Identity Evaluations or Undetermined Coefficients to find a particular solution $w_P(t)$. Then a general solution will be given by $w(t) = w_H(t) + w_P(t)$.

Zero Degree Formula. Because $d = 0$, $\mu + i\nu = i4$, and $m = 0$, we may use the zero degree formula with $m = 0$. Because $p(z) = z^2 - 5z + 6$, we see that $p(i4) = (i4)^2 - 5\cdot i4 + 60 = -16 - i20 + 6 = -(10 + i20)$, and that

$$\mathrm{L}\left(\frac{e^{i4t}}{p(i4)}\right) = \mathrm{L}\left(\frac{e^{i4t}}{-(10+i20)}\right) = e^{i4t}\,.$$

Because $\mathrm{L}(w) = 20\sin(4t) = 20\,\mathrm{Im}(e^{i4t})$, we see that a particular solution is

$$\begin{aligned} w_P(t) &= -\,\mathrm{Im}\left(\frac{20e^{i4t}}{10+i20}\right) = -2\,\mathrm{Im}\left(\frac{e^{i4t}}{1+i2}\,\frac{1-i2}{1-i2}\right) = -2\,\mathrm{Im}\left(\frac{(1-i2)e^{i4t}}{1^2+2^2}\right) \\ &= -\tfrac{2}{5}\,\mathrm{Im}\big((1-i2)(\cos(4t)+i\sin(4t))\big) = -\tfrac{2}{5}\big(-2\cos(4t)+\sin(4t)\big) \\ &= \tfrac{4}{5}\cos(2t) - \tfrac{2}{5}\sin(4t)\,. \end{aligned}$$

Therefore a general solution is

$$w(t) = c_1 e^{2t} + c_2 e^{3t} + \tfrac{4}{5}\cos(2t) - \tfrac{2}{5}\sin(4t)\,.$$

Key Indentity Evaluations. Because $m = m + d = 0$ for the forcing $20\sin(4t)$, we need only the Key Identity

$$\mathrm{L}\big(e^{zt}\big) = p(z)e^{zt} = (z^2 - 5z + 6)e^{zt}\,.$$

By evaluating this at the characteristic $z = i4$ we obtain

$$\mathrm{L}\big(e^{i4t}\big) = \big((i4)^2 - 5(i4) + 6\big)e^{i4t} = (-16 - i20 + 6)e^{i4t} = -(10 + i20)e^{i4t}\,.$$

Because $\mathrm{L}(w) = 20\sin(4t) = 20\,\mathrm{Im}(e^{i4t})$, we see that a particular solution is

$$\begin{aligned} w_P(t) &= -\,\mathrm{Im}\left(\frac{20e^{i4t}}{10+i20}\right) = -2\,\mathrm{Im}\left(\frac{e^{i4t}}{1+i2}\,\frac{1-i2}{1-i2}\right) = -2\,\mathrm{Im}\left(\frac{(1-i2)e^{i4t}}{1^2+2^2}\right) \\ &= -\tfrac{2}{5}\,\mathrm{Im}\big((1-i2)(\cos(4t)+i\sin(4t))\big) = -\tfrac{2}{5}\big(-2\cos(4t)+\sin(4t)\big) \\ &= \tfrac{4}{5}\cos(2t) - \tfrac{2}{5}\sin(4t)\,. \end{aligned}$$

Therefore a general solution is

$$w(t) = c_1 e^{2t} + c_2 e^{3t} + \tfrac{4}{5}\cos(2t) - \tfrac{2}{5}\sin(4t)\,.$$

Undetermined Coefficients. Because $\mu + i\nu = i4$ and $m = m + d = 0$ for the forcing $20\sin(4t)$, we *seek* a particular solution in the form

$$w_P(t) = A\cos(4t) + B\sin(4t)\,.$$

Because

$$w_P'(t) = -4A\sin(4t) + 4B\cos(4t)\,, \qquad w_P''(t) = -16A\cos(4t) - 16B\sin(4t)\,,$$

we see that

$$\begin{aligned} \mathrm{L}w_P(t) &= w_P''(t) - 5w_P'(t) + 6w_P(t) \\ &= \big[-16A\cos(4t) - 16B\sin(4t)\big] - 5\big[-4A\sin(4t) + 4B\cos(4t)\big] \\ &\quad + 6\big[A\cos(4t) + B\sin(4t)\big] \\ &= (-10A - 20B)\cos(4t) + (20A - 10B)\sin(4t)\,. \end{aligned}$$

By setting $\mathrm{L}w_P(t) = 20\sin(4t)$, we see that

$$-10A - 20B = 0\,, \qquad 20A - 10B = 20\,.$$

The first equation implies $A = -2B$, which when placed into the second equation yields $-50B = 20$. Hence, $B = -\frac{2}{5}$ and $A = \frac{4}{5}$, which gives

$$w_P(t) = \tfrac{4}{5}\cos(4t) - \tfrac{2}{5}\sin(4t)\,.$$

Therefore a general solution is

$$w(t) = c_1 e^{2t} + c_2 e^{3t} + \tfrac{4}{5}\cos(4t) - \tfrac{2}{5}\sin(4t)\,.$$

(6) [8] What answer will be produced by the following Matlab commands?

```
>> ode = 'D2y − 8*Dy + 20*y = 20*exp(5*t)';
>> dsolve(ode, 't')
```

ans =

Solution. The commands ask Matlab to give a general solution of the equation

$$\mathrm{D}^2 y - 8\mathrm{D}y + 20y = 20e^{5t}\,, \qquad \text{where} \quad \mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}t}\,.$$

This is a nonhomogeneous linear equation with constant coefficients. Its characteristic polynomial is

$$p(z) = z^2 - 8z + 20 = (z-4)^2 + 4 = (z-4)^2 + 2^2\,,$$

which has the conjugate pair of roots $4 \pm i2$. Therefore a general solution of the associated homogeneous equation is

$$y_H(t) = c_1 e^{4t}\cos(2t) + c_2 e^{4t}\sin(2t)\,.$$

The forcing $20e^{5t}$ has characteristic form with degree $d = 0$ and characteristic $\mu + i\nu = 5$, which is a root of $p(z)$ of multiplicity $m = 0$. Therefore we can use either Key Identity Evaluations or Undetermined Coefficients to find a particular solution $y_P(t)$.

Zero Degree Formula. Because $d = 0$, $\mu + i\nu = 5$, and $m = 0$, we may use the zero degree formula with $m = 0$. Because $p(z) = z^2 - 8z + 20$, we see that $p(5) = 5^2 - 8\cdot 5 + 20 = 25 - 40 + 20 = 5$, and that

$$\mathrm{L}\left(\frac{e^{5t}}{p(5)}\right) = \mathrm{L}\left(\frac{e^{5t}}{5}\right) = e^{5t}\,.$$

Because $\mathrm{L}(y) = 20e^{5t}$, we see that a particular solution is $y_P(t) = 4e^{5t}$. Therefore a general solution is

$$y(t) = c_1 e^{4t}\cos(2t) + c_2 e^{4t}\sin(2t) + 4e^{5t}\,.$$

Remark. Had you forgotten the zero degree formula then you could have derived it by Key Identity Evaluations as in the following solution.

Key Indentity Evaluations. Because $m = m + d = 0$ we only need the Key identity, which is

$$\mathrm{L}\big(e^{zt}\big) = p(z)e^{zt} = (z^2 - 8z + 20)e^{zt}\,.$$

By evaluating this at the characteristic $z = 5$ we obtain

$$\mathrm{L}\big(e^{5t}\big) = (5^2 - 8\cdot 5 + 20)e^{5t} = (25 - 40 + 20)e^{5t} = 5e^{5t}\,.$$

Because $\mathrm{L}(y) = 20e^{5t}$, we see that a particular solution is $y_P(t) = 4e^{5t}$. Therefore a general solution is

$$y(t) = c_1 e^{4t}\cos(2t) + c_2 e^{4t}\sin(2t) + 4e^{5t}\,.$$

Undetermined Coefficients. Because $\mu + i\nu = 5$ and $m = m + d = 0$ for the forcing $20e^{5t}$, we *seek* a particular solution in the form

$$y_P(t) = Ae^{5t}\,.$$

Because $y_P'(t) = 5Ae^{5t}$ and $y_P''(t) = 25Ae^{5t}$, we see that

$$\begin{aligned}\mathrm{L}y_P(t) &= y_P''(t) - 8y_P'(t) + 20y_P(t)\\ &= \big[25Ae^{5t}\big] - 8\big[5Ae^{5t}\big] + 20\big[Ae^{5t}\big]\\ &= (25 - 40 + 20)Ae^{5t} = 5Ae^{5t}\,.\end{aligned}$$

By setting $\mathrm{L}y_P(t) = 20e^{5t}$, we see that $5A = 20$, whereby $A = 4$. Hence, we obtain the particular solution $y_P(t) = 4e^{5t}$. Therefore a general solution is

$$y(t) = c_1e^{4t}\cos(2t) + c_2e^{4t}\sin(2t) + 4e^{5t}\,.$$

(7) [8] Compute the Green function $g(t)$ associated with the differential operator

$$\mathrm{D}^2 + 8\mathrm{D} + 16\,, \qquad \text{where} \quad \mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}t}\,.$$

Solution. Because the differential operator has constant coefficients, the Green function $g(t)$ associated with it satisfies the initial-value problem

$$\mathrm{D}^2g + 8\mathrm{D}g + 16g = 0\,, \qquad g(0) = 0\,, \quad g'(0) = 1\,.$$

The characteristic polynomial is

$$p(z) = z^2 + 8z + 16 = (z + 4)^2\,,$$

which has the double real root -4. Hence, the Green function has the form

$$g(t) = c_1e^{-4t} + c_2t\,e^{-4t}\,.$$

The initial condition $g(0) = 0$ implies that $c_1 = 0$. Because

$$g'(t) = -4c_2t\,e^{-4t} + c_2e^{-4t}\,,$$

the initial condition $g'(0) = 1$ implies that $c_2 = 1$. Therefore the Green function is

$$g(t) = t\,e^{-4t}\,.$$

(8) [9] Solve the initial-value problem

$$w'' + 8w' + 16w = \frac{8e^{-4t}}{1 + 4t^2}\,, \qquad w(0) = w'(0) = 0\,.$$

Solution. By the last problem the Green function for this problem is $g(t) = t\,e^{-4t}$. Because this equation is in normal form and because both the initial values are 0, the solution to this inital-value problem is given by the Green function formula

$$\begin{aligned}w(t) &= \int_0^t g(t - s)f(s)\,\mathrm{d}s = \int_0^t (t - s)\,e^{-4(t-s)}\,\frac{8e^{-4s}}{1 + 4s^2}\,\mathrm{d}s\\ &= t\,e^{-4t}\int_0^t \frac{8}{1 + 4s^2}\,\mathrm{d}s - e^{-4t}\int_0^t \frac{8s}{1 + 4s^2}\,\mathrm{d}s\,.\end{aligned}$$

Because

$$\int_0^t \frac{8}{1+4s^2}\,\mathrm{d}s = 4\tan^{-1}(2s)\Big|_{s=0}^{t} = 4\tan^{-1}(2t)\,,$$
$$\int_0^t \frac{8s}{1+4s^2}\,\mathrm{d}s = \log(1+4s^2)\Big|_{s=0}^{t} = \log(1+4t^2)\,,$$

we find that

$$w(t) = 4t\,e^{-4t}\tan^{-1}(2t) - e^{-4t}\log(1+4t^2)\,.$$

Remark. This problem can also be solved by using variation of parameters. However that approach is not as efficient because it does not directly solve the initial-value problem. Rather, after finding a particular solution the constants c_1 and c_2 in $W_H(t)$ must be determined to satisfy the initial conditions.

(9) [10] The functions $1-t$ and e^{-t} are solutions of the homogeneous equation

$$t\,x'' - (1-t)x' - x = 0 \qquad \text{over } t > 0\,.$$

(You do not have to check that this is true!)
(a) [3] Show that these functions are linearly independent.
(b) [7] Give a general solution of the nonhomogeneous equation

$$t\,y'' - (1-t)y' - y = \frac{t^2}{1-t} \qquad \text{over } t > 0\,.$$

Solution (a). The Wronskian of $1-t$ and e^{-t} is

$$\begin{aligned} W\big[1-t, e^{-t}\big](t) &= \det\begin{pmatrix} 1-t & e^{-t} \\ -1 & -e^{-t} \end{pmatrix} = (1-t)\cdot(-e^{-t}) - (-1)\cdot e^{-t} \\ &= -e^{-t} + t\,e^{-t} + e^{-t} = t\,e^{-t}\,. \end{aligned}$$

Because $W[1-t, e^{-t}](t) = t\,e^{-t} \neq 0$ for $t > 0$, the functions $1-t$ and e^{-t} are linearly independent.

Solution (b). Because $1-t$ and e^{-t} are linearly independent, a general solution of the associated homogeneous problem is

$$y_H(t) = c_1(1-t) + c_2 e^{-t}\,.$$

Because this problem has variable coefficients, we should use either the general Green Function method or Variation of Parameters to find a particular solution $y_P(t)$. Both of these methods require the equation to be put into its normal form, which is

$$y'' - \frac{1-t}{t}\,y' - \frac{1}{t}\,y = \frac{t}{1-t}\,.$$

Notice that the forcing is not defined at $t = 1$.

General Green Function. The Green function $G(t,s)$ is given by

$$G(t,s) = \frac{1}{W[1-s, e^{-s}](s)}\,\det\begin{pmatrix} 1-s & e^{-s} \\ 1-t & e^{-t} \end{pmatrix} = \frac{(1-s)e^{-t} - (1-t)e^{-s}}{s\,e^{-s}}\,.$$

The Green function formula with any $t_I > 0$ such that $t_I \neq 1$ then yields the solution

$$\begin{aligned} y_P(t) &= \int_{t_I}^{t} G(t,s)\, f(s)\, \mathrm{d}s = \int_{t_I}^{t} \frac{(1-s)e^{-t} - (1-t)e^{-s}}{s\, e^{-s}} \, \frac{s}{1-s}\, \mathrm{d}s \\ &= e^{-t} \int_{t_I}^{t} e^{s}\, \mathrm{d}s - (1-t) \int_{t_I}^{t} \frac{1}{1-s}\, \mathrm{d}s \,. \end{aligned}$$

We can evaluate the above definite integrals as

$$\int_{t_I}^{t} e^{s}\, \mathrm{d}s = e^{s} \Big|_{s=t_I}^{t} = e^{t} - e^{t_I} \,,$$

$$-\int_{t_I}^{t} \frac{1}{1-s}\, \mathrm{d}s = \log(|1-s|) \Big|_{s=t_I}^{t} = \log(|1-t|) - \log(|1-t_I|) = \log\left(\left| \frac{1-t}{1-t_I} \right| \right) .$$

Therefore a general solution is

$$y(t) = c_1(1-t) + c_2 e^{-t} + 1 - e^{t_I - t} + (1-t)\log\left(\left| \frac{1-t}{1-t_I} \right| \right) .$$

Variation of Parameters. We *seek* a solution in the form

$$y(t) = u_1(t)(1-t) + u_2(t)e^{-t} \,.$$

where $u_1'(t)$ and $u_2'(t)$ satisfy the linear algebraic system

$$u_1'(t)(1-t) + u_2'(t)e^{-t} = 0 \,, \qquad -u_1'(t) - u_2'(t)e^{-t} = \frac{t}{1-t} \,.$$

The solution of this system is

$$u_1'(t) = -\frac{1}{1-t} \,, \qquad u_2'(t) = e^{t} \,.$$

Alternatively, because $W[1-t, e^{-t}](t) = t\, e^{-t}$, the formulas from the notes yield

$$u_1'(t) = -\frac{e^{-t}}{t\, e^{-t}} \, \frac{t}{1-t} = -\frac{1}{1-t} \,, \qquad u_2'(t) = \frac{1-t}{t\, e^{-t}} \, \frac{t}{1-t} = e^{t} \,.$$

No matter how they are obtained, you integrate these equations to find

$$\begin{aligned} u_1(t) &= -\int \frac{1}{1-t}\, \mathrm{d}t = c_1 + \log(|1-t|) \,, \\ u_2(t) &= \int e^{t}\, \mathrm{d}t = c_2 + e^{t} \,. \end{aligned}$$

Therefore a general solution is

$$\begin{aligned} y(t) &= \big(c_1 + \log(|1-t|)\big)(1-t) + \big(c_2 + e^{t}\big)e^{-t} \\ &= c_1(1-t) + c_2 e^{-t} + 1 + (1-t)\log(|1-t|) \,. \end{aligned}$$

Remark. This general solution appears different than the one obtained by the general Green function method. However, replacing c_1 and c_1 in this solution with $c_1 - \log(|1-t_I|)$ and $c_2 - e^{t_I}$ transforms into the earlier one, so they are equivalent.

(10) [7] Find a particular solution $v_P(t)$ of the equation $v'' - 16v = 32e^{-4t}$.

Solution. This is a nonhomogeneous linear equation with constant coefficients. Its characteristic polynomial is

$$p(z) = z^2 - 16 = (z+4)(z-4)\,,$$

which has two simple real roots -4 and 4. The forcing $32e^{-4t}$ has characteristic form with degree $d = 0$ and characteristic $\mu + i\nu = -4$, which is a root of $p(z)$ of multiplicity $m = 1$. Therefore we can use either Key Identity Evaluations or Undetermined Coefficients to find a particular solution $v_P(t)$.

Zero Degree Formula. Because $d = 0$, $\mu + i\nu = -4$, and $m = 1$, we may use the zero degree formula with $m = 1$. Because $p(z) = z^2 - 16$, we see that $p'(z) = 2z$, that $p'(-4) = -8$, and that

$$\mathrm{L}\left(\frac{t\,e^{-4t}}{p'(-4)}\right) = \mathrm{L}\left(\frac{t\,e^{-4t}}{-8}\right) = e^{-4t}\,.$$

Because $\mathrm{L}(v) = 32e^{-4t}$, we see that a particular solution is $v_P(t) = -4t\,e^{-4t}$.

Remark. Had you forgotten the zero degree formula then you could have derived it by Key Identity Evaluations as in the following solution.

Key Indentity Evaluations. Because $m = m + d = 1$ for the forcing $32e^{-4t}$, we only need the first derivative of the Key Identity. The Key Identity and its first derivative are

$$\mathrm{L}\big(e^{zt}\big) = (z^2 - 16)e^{zt}\,,$$
$$\mathrm{L}\big(t\,e^{zt}\big) = (z^2 - 16)t\,e^{zt} + 2ze^{zt}\,.$$

By evaluating the first derivative of the Key identity at the characteristic $z = -4$ we obtain

$$\mathrm{L}\big(t\,e^{-4t}\big) = 2\cdot(-4)\cdot e^{-4t} = -8e^{-4t}\,.$$

Because $\mathrm{L}(v) = 32e^{-4t}$, we see that a particular solution is $v_P(t) = -4t\,e^{-4t}$.

Undetermined Coefficients. Because $\mu + i\nu = 3$ and $m = m + d = 1$ for the forcing $32e^{-4t}$, we *seek* a particular solution in the form

$$v_P(t) = At\,e^{-4t}\,.$$

Because

$$v_P'(t) = -4At\,e^{-4t} + Ae^{-4t}\,, \qquad v_P''(t) = 16At\,e^{-4t} - 8Ae^{-4t}\,,$$

we obtain

$$\mathrm{L}v_P(t) = \big[16At\,e^{-4t} - 8Ae^{-4t}\big] - 16\big[At\,e^{-4t}\big] = -8Ae^{-4t}\,.$$

By setting $\mathrm{L}v_P(t) = 32e^{-4t}$, we see that $-8A = 32$, whereby $A = -4$. Therefore, a particular solution is $v_P(t) = -4t\,e^{-4t}$.

Remark. A general solution is $v(t) = c_1e^{4t} + c_2e^{-4t} - 4t\,e^{-4t}$.

(11) [9] The vertical displacement of an unforced mass on a spring is given by

$$h(t) = -4e^{-5t}\cos(12t) - 3e^{-5t}\sin(12t)\,.$$

(a) [2] Is this system undamped, under damped, critically damped, or over damped? (Give your reasoning!)
(b) [5] Express $h(t)$ in the amplitude-phase form $h(t) = Ae^{-5t}\cos(12t - \delta)$ with $A > 0$ and $0 \leq \delta < 2\pi$. Label the amplitude and phase. (The phase may be expressed in terms of an inverse trig function.)
(c) [2] Give the natural frequency and natural period of this spring-mass system.

Solution (a). The system is *under damped* because the vertical displacement $h(t)$ arises from a characteristic polynomial with the conjugate pair of roots $-5 \pm i12$.

Solution (b). By comparing

$$Ae^{-5t}\cos(12t - \delta) = Ae^{-5t}\cos(\delta)\cos(12t) + Ae^{-5t}\sin(\delta)\sin(12t)\,,$$

with $h(t) = -4e^{-5t}\cos(12t) - 3e^{-5t}\sin(12t)$, we see that

$$A\cos(\delta) = -4\,, \qquad A\sin(\delta) = -3\,.$$

This shows that (A, δ) are the polar coordinates of the point in the plane whose Cartesian coordinates are $(-4, -3)$. Clearly A is given by

$$A = \sqrt{(-4)^2 + (-3)^2} = \sqrt{16 + 9} = \sqrt{25} = 5\,.$$

Because $(-4, -3)$ lies in the *third quadrant*, the phase δ must satisfy $\pi < \delta < \frac{3}{2}\pi$. We can express δ several ways. A picture shows that if we use π as a reference then

$$\cos(\delta - \pi) = \tfrac{4}{5}\,, \qquad \sin(\delta - \pi) = \tfrac{3}{5}\,, \qquad \tan(\delta - \pi) = \tfrac{3}{4}\,,$$

whereby we can express the phase by any one of the formulas

$$\delta = \pi + \cos^{-1}\big(\tfrac{4}{5}\big)\,, \qquad \delta = \pi + \sin^{-1}\big(\tfrac{3}{5}\big)\,, \qquad \delta = \pi + \tan^{-1}\big(\tfrac{3}{4}\big)\,.$$

The same picture shows that if we use $\frac{3}{2}\pi$ as a reference then

$$\cos\big(\tfrac{3}{2}\pi - \delta\big) = \tfrac{3}{5}\,, \qquad \sin\big(\tfrac{3}{2}\pi - \delta\big) = \tfrac{4}{5}\,, \qquad \tan\big(\tfrac{3}{2}\pi - \delta\big) = \tfrac{4}{3}\,,$$

whereby we can express the phase by any one of the formulas

$$\delta = \tfrac{3}{2}\pi - \cos^{-1}\big(\tfrac{3}{5}\big)\,, \qquad \delta = \tfrac{3}{2}\pi - \sin^{-1}\big(\tfrac{4}{5}\big)\,, \qquad \delta = \tfrac{3}{2}\pi - \tan^{-1}\big(\tfrac{4}{3}\big)\,.$$

Only one expression for δ is required.

Remark. It is incorrect to give the phase by one of the formulas

$$\delta = \cos^{-1}\big(-\tfrac{4}{5}\big)\,, \qquad \delta = \sin^{-1}\big(-\tfrac{3}{5}\big)\,, \qquad \delta = \tan^{-1}\big(\tfrac{3}{4}\big)\,,$$

because, by our conventions for the range of the inverse trigonometric functions, $\cos^{-1}\big(-\frac{4}{5}\big)$ lies in $(\frac{\pi}{2}, \pi)$, $\sin^{-1}\big(-\frac{3}{5}\big)$ lies in $(-\frac{\pi}{2}, 0)$, and $\tan^{-1}\big(\frac{3}{4}\big)$ lies in $(0, \frac{\pi}{2})$.

Solution (c). Because the underlying characteristic polynomial has the conjugate pair of roots $-5 \pm i12$, it must be

$$p(z) = (z + 5)^2 + 12^2 = z^2 + 10z + 25 + 144 = z^2 + 10z + 169\,.$$

Therefore the vertical displacement $h(t)$ satisfies the differential equation

$$\ddot{h} + 10\dot{h} + 169h = 0\,.$$

We can read off that the natural frequency is $\omega_o = \sqrt{169} = 13$ radians per sec, whereby the natural period T_o is given by

$$T_o = \frac{2\pi}{\omega_o} = \frac{2\pi}{13} \quad \text{sec}\,.$$

(12) [8] When a 10 gram mass is hung vertically from a spring, at rest it stretches the spring 5.0 cm. (Gravitational acceleration is $g = 980$ cm/sec^2.) At $t = 0$ the mass is displaced 7 cm below its rest position and is released with a downward velocity of 3 cm/sec. The medium imparts a damping force of 900 dynes (1 dyne = 1 gram cm/sec^2) when the speed of the mass is 4 cm/sec. Assume that the spring force is proportional to displacement, that the damping is proportional to veloicity, and that there are no other forces.

(a) [6] Formulate an initial-value problem that governs the motion of the mass for $t > 0$. (DO NOT solve this initial-value problem, just write it down!)

(b) [2] Is this system undamped, under damped, critically damped, or over damped? (Give your reasoning!)

Solution (a). Let $h(t)$ be the displacement (in centimeters) of the mass from its rest position at time t (in seconds), with upward displacements being positive. The governing initial-value problem then has the form

$$m\ddot{h} + \gamma\dot{h} + kh = 0\,, \qquad h(0) = -7\,, \quad h'(0) = -3\,,$$

where m is the mass, γ is the damping coefficient, and k is the spring constant. We are given that $m = 10$ grams. We obtain k by balancing the force applied by the spring when it is stetched 5.0 cm with the weight of the mass ($mg = 10 \cdot 980$ dynes). This gives $k\,5.0 = 10 \cdot 980$, or

$$k = \frac{10 \cdot 980}{5.0} = 2 \cdot 980 \text{ dynes/cm}\,.$$

We obtain γ by balancing the damping force when the speed of the mass is 4 cm/sec with 900 dynes. This gives $\gamma\,4 = 900$, or

$$\gamma = \frac{900}{4} \quad \text{dynes sec/cm}\,.$$

Therefore the governing initial-value problem is

$$10\ddot{h} + \tfrac{900}{4}\dot{h} + 2 \cdot 980h = 0\,, \qquad h(0) = -7\,, \quad \dot{h}(0) = -3\,.$$

Remark. Had we chosen the convention of downward displacements being positive then the governing initial-value problem is

$$10\ddot{h} + \tfrac{900}{4}\dot{h} + 2 \cdot 980h = 0\,, \qquad h(0) = 7\,, \quad \dot{h}(0) = 3\,.$$

Solution (b). The normal form of the governing equation is

$$\ddot{h} + \tfrac{90}{4}\dot{h} + 2 \cdot 98h\,.$$

Its charactericitc polynomial is

$$p(z) = z^2 + \tfrac{90}{4}z + 196 = \left(z + \tfrac{45}{4}\right)^2 + 196 - \left(\tfrac{45}{4}\right)^2.$$

Because $196 - (\frac{45}{4})^2 > 0$, this polynomial has a conjugate pair of roots. Therefore the system is *under damped.*