

Final Exam, Math 246/246H
Monday, 14 December 2015

Closed book. No electronics. Answer only one question on each answer page. Write your name and which question is being answered on each answer page. Sign the Honor Pledge on the first answer page only. Indicate your answer to each part of each question clearly. Give reasoning that justifies your answers.

- (1) [22] Give the explicit solution to each of the following initial-value problems. Identify their intervals of definition.

(a) [11] $u' = e^{-2t}u^3$, $u(0) = -1$.
(b) [11] $(1+t)x' + x = e^t$, $x(0) = 3$.

- (2) [12] Consider the following Matlab function m-file.

```
function [t,y] = solveit(tI, yI, tF, n)
t = zeros(n + 1, 1); y = zeros(n + 1, 1);
t(1) = tI; y(1) = yI; h = (tF - tI)/n; hhalf = h/2;
for i = 1:n
thalf = t(i) + hhalf; yhalf = y(i) + hhalf*(t(i)^2 - y(i)^2);
t(i + 1) = t(i) + h; y(i + 1) = y(i) + h*(thalf^2 - yhalf^2);
end
```

Suppose the input values are $tI = 1$, $yI = 0$, $tF = 9$, and $n = 40$.

- (a) [4] What initial-value problem is being approximated numerically?
(b) [1] What numerical method is being used?
(c) [1] What is the step size?
(d) [6] What are the output values of $t(2)$ and $y(2)$?
- (3) [22] Give an explicit real-valued general solution to each of the following equations.
- (a) [11] $v'' + 6v' + 25v = 8e^{-3t}$
(b) [11] $h'' + 4h' - 5h = 12e^t$
- (4) [12] Given the fact that $1 + t$ and e^t are solutions of the associated homogeneous differential equation, solve the initial-value problem

$$tw'' - (1+t)w' + w = \frac{t^2}{1+t}, \quad w(1) = 0, \quad w'(1) = 0.$$

Evaluate any definite integrals that arise.

- (5) [22] Let $y(t)$ be the solution of the initial-value problem

$$y'' - 6y' + 10y = f(t), \quad y(0) = 4, \quad y'(0) = -7,$$

where $f(t) = t^2 + u(t-2)(4-t^2)$. Here u is the unit step function.

- (a) [11] Find the Laplace transform $F(s)$ of the forcing $f(t)$.
(b) [11] Find the Laplace transform $Y(s)$ of the solution $y(t)$.
(DO NOT take the inverse Laplace transform to find $y(t)$; just solve for $Y(s)$!)

You may refer to the table.

More Problems on the Other Side!

- (6) [12] Find the function $x(t)$ whose Laplace transform is given by $X(s) = \frac{e^{-4s}(s+11)}{(s-7)(s+2)}$.
You may refer to the table.

- (7) [9] Two interconnected tanks are filled with brine (salt water). At $t = 0$ the first tank contains 55 liters and the second contains 36 liters. Brine with a salt concentration of 8 grams per liter flows into the first tank at 4 liters per hour. Well-stirred brine flows from the first tank into the second at 7 liters per hour, from the second into the first at 6 liters per hour, from the first into a drain at 3 liter per hour, and from the second into a drain at 5 liters per hour. At $t = 0$ there are 17 grams of salt in the first tank and 23 grams in the second. Give an initial-value problem that governs the amount of salt in each tank as a function of time.

- (8) [9] What answer will be produced by the following Matlab command?

$$\gg A = [3 \ 2; 5 \ 0]; [\text{vect}, \text{val}] = \text{eig}(\text{sym}(A))$$

You do not have to give the answer in Matlab format.

- (9) [12] Compute the 2×2 matrix $e^{t\mathbf{B}}$ for the matrix \mathbf{B} with eigenpairs

$$\left(3, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right), \quad \left(2, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right).$$

- (10) [12] Solve the initial-value problem $\mathbf{x}' = \mathbf{A}\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}^I$ where

$$\mathbf{A} = \begin{pmatrix} 3 & -5 \\ 2 & 1 \end{pmatrix}, \quad \mathbf{x}^I = \begin{pmatrix} 3 \\ 0 \end{pmatrix}.$$

- (11) [12] Consider the nonlinear planar system

$$p' = p(2 - q), \quad q' = q(p - 5).$$

- (a) [6] Find all of its stationary points.
(b) [6] Find all of its semistationary solutions.

- (12) [22] Consider the nonlinear planar system

$$x' = y - \frac{1}{2}x^2 + 8, \quad y' = xy.$$

- (a) [11] Find a nonconstant function $H(x, y)$ such that every orbit of the system satisfies $H(x, y) = c$ for some constant c .
(b) [11] The stationary points are $(-4, 0)$, $(4, 0)$, and $(0, -8)$. In the phase-plane sketch these stationary points plus the level set $H(x, y) = c$ for each value of c that corresponds to a stationary point that is a saddle. (No arrows are required!)

- (13) [22] Consider the nonlinear planar system

$$u' = -3u + v, \quad v' = -2u - v - 5u^2.$$

Its stationary points are $(0, 0)$ and $(-1, -3)$.

- (a) [6] Find the Jacobian matrix at each stationary point.
(b) [8] Classify the type and stability of each stationary point.
(c) [8] Sketch a phase-plane portrait of the system that shows its behavior near each stationary point. (Carefully mark all sketched orbits with arrows!)