

Solutions of Sample Problems for First In-Class Exam
Math 246, Fall 2015, Professor David Levermore

- (1) (a) Give the integral being evaluated by the following Matlab command.

```
int('x/(1+x^4)', 'x', 0, inf)
```

Solution. It is evaluating the definite integral

$$\int_0^{\infty} \frac{r}{1+r^4} dr.$$

where you can replace r by any other variable.

- (b) Sketch the graph that would be produced by the following Matlab command.

```
ezplot('2/t', [1, 6])
```

Solution. Your sketch should show a decreasing, concave up function that decreases from a value of 2 to a value of $\frac{1}{3}$ over the interval $[1, 6]$.

- (c) Sketch the graph that would be produced by the following Matlab commands.

```
[X, Y] = meshgrid(-5:0.1:5, -5:0.1:5)
contour(X, Y, X.^2 + Y.^2, [1, 9, 25])
axis square
```

Solution. Your sketch should show both x and y axes marked from -5 to 5 and circles of radius 1, 3, and 5 centered at the origin.

- (2) Find the explicit solution for each of the following initial-value problems and identify its interval of definition.

(a) $\frac{dz}{dt} = \frac{\cos(t) - z}{1+t}, \quad z(0) = 2.$

Solution. This equation is *linear* in z , so write it in the linear normal form

$$\frac{dz}{dt} + \frac{z}{1+t} = \frac{\cos(t)}{1+t}.$$

An integrating factor is given by

$$\exp\left(\int_0^t \frac{1}{1+s} ds\right) = \exp(\log(1+t)) = 1+t,$$

Upon multiplying the normal form of the equation by $(1+t)$, we obtain the integrating factor form

$$\frac{d}{dt}((1+t)z) = \cos(t).$$

This is then integrated to obtain

$$(1+t)z = \sin(t) + c.$$

The integration constant c is found through the initial condition $z(0) = 2$ by setting $t = 0$ and $z = 0$, whereby

$$c = (1 + 0)2 - \sin(0) = 2.$$

Hence, upon solving explicitly for z , the solution is

$$z = \frac{2 + \sin(t)}{1 + t}.$$

The interval of definition for this solution is $t > -1$.

(b) $\frac{du}{dz} = e^u + 1, \quad u(0) = 0.$

Solution. This equation is *autonomous* (and therefore *separable*). Its separated differential form is

$$\frac{1}{e^u + 1} du = dz.$$

This equation can be integrated to obtain

$$z = \int \frac{1}{e^u + 1} du = \int \frac{e^{-u}}{1 + e^{-u}} du = -\log(1 + e^{-u}) + c.$$

The integration constant c is found through the initial condition $u(0) = 0$ by setting $z = 0$ and $u = 0$, whereby

$$c = 0 + \log(1 + e^0) = \log(2).$$

Hence, the solution is given implicitly by

$$z = -\log(1 + e^{-u}) + \log(2) = -\log\left(\frac{1 + e^{-u}}{2}\right).$$

This may be solve for u as follows:

$$\begin{aligned} e^{-z} &= \frac{1 + e^{-u}}{2}, \\ 2e^{-z} - 1 &= e^{-u}, \\ u &= -\log(2e^{-z} - 1). \end{aligned}$$

The interval of definition for this solution is $z < \log(2)$.

(c) $\frac{dv}{dt} = -3t^2e^{-v}, \quad v(2) = 0.$

Solution. This equation is separable. Its separated differential form is

$$e^v dv = -3t^2 dt.$$

This can be integrated to obtain

$$e^v = -t^3 + c.$$

The initial condition $v(2) = 0$ implies that $c = e^0 + 2^3 = 1 + 8 = 9$. Therefore $e^v = -t^3 + 9$, which can be solved as

$$v = \log(9 - t^3), \quad \text{with interval of definition } t < 9^{\frac{1}{3}}.$$

Here we need $9 > t^3$ for the log to be defined. The interval of definition is obtained by taking the cube root of both sides of this inequality.

(3) Consider the differential equation

$$\frac{dy}{dt} = 4y^2 - y^4.$$

(a) Find all of its stationary solutions and classify each as being either stable, unstable, or semistable.

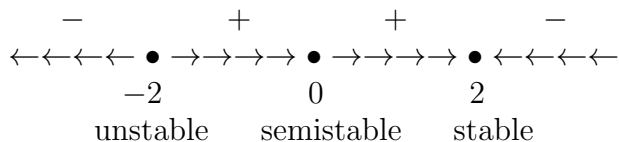
Solution. The right-hand side of the equation factors as

$$4y^2 - y^4 = y^2(4 - y^2) = y^2(2 + y)(2 - y),$$

which implies that $y = -2$, $y = 0$, and $y = 2$ are all of its stationary solutions. A sign analysis of $y^2(2 + y)(2 - y)$ then shows that

$$\begin{aligned} \frac{dy}{dt} &> 0 && \text{when } -2 < y < 0 \text{ or } 0 < y < 2, \\ \frac{dy}{dt} &< 0 && \text{when } -\infty < y < -2 \text{ or } 2 < y < \infty. \end{aligned}$$

Therefore the phase-line portrait for this equation is



(b) If $y(-1) = 1$, how does the solution $y(t)$ behave as $t \rightarrow \infty$?

Solution. It is clear from the answer to (a) that

$$\frac{dy}{dt} > 0 \quad \text{when } 0 < y < 2,$$

so that $y(t) \rightarrow 2$ as $t \rightarrow \infty$ if $y(-1) = 1$.

(c) If $y(3) = -1$, how does the solution $y(t)$ behave as $t \rightarrow \infty$?

Solution. It is clear from the answer to (a) that

$$\frac{dy}{dt} > 0 \quad \text{when } -2 < y < 0,$$

so that $y(t) \rightarrow 0$ as $t \rightarrow \infty$ if $y(3) = -1$.

(d) Sketch a graph of y versus t showing the direction field and several solution curves. The graph should show all the stationary solutions as well as solution curves above and below each of them. Every value of y should lie on at least one sketched solution curve.

Solution. Will be given during the review session.

- (4) Give the interval of definition for the solution of the initial-value problem

$$\frac{dx}{dt} + \frac{1}{t^2 - 4}x = \frac{1}{\sin(t)}, \quad x(1) = 0.$$

(You do not have to solve this equation to answer this question!)

Solution. This problem is linear in x and is already in normal form. The coefficient $1/(t^2 - 4)$ is continuous everywhere except where $t = \pm 2$, at which it is undefined. The forcing $1/\sin(t)$ is continuous everywhere except where $t = n\pi$ for some integer n — i.e. everywhere except where $t = 0, \pm\pi, \pm 2\pi, \dots$, at which it is undefined. The initial time is $t = 1$. Therefore we can read off that the interval of definition is $(0, 2)$ because

- the initial time $t = 1$ is in $(0, 2)$,
- both the coefficient and forcing are continuous over $(0, 2)$,
- the forcing is undefined at $t = 0$,
- the coefficient is undefined at $t = 2$.

- (5) In the absence of predators the population of mosquitoes in a certain area would double every three weeks. There are 120,000 mosquitoes in the area initially, and predators eat 90,000 mosquitoes per week. Write down an initial-value problem that governs the population of mosquitoes in the area at any time. (You do not have to solve the initial-value problem!)

Solution. Let $M(t)$ be the number of mosquitoes at time t weeks. Doubling every three weeks corresponds to a growth factor of $2^{\frac{t}{3}} = (e^{\log(2)})^{\frac{t}{3}} = e^{\frac{1}{3}\log(2)t}$, which implies a growth rate of $\frac{1}{3}\log(2)$. Therefore the initial-value problem that M satisfies is

$$\frac{dM}{dt} = \frac{1}{3}\log(2)M - 90,000, \quad M(0) = 120,000.$$

- (6) A tank initially contains 100 liters of pure water. Beginning at time $t = 0$ brine (salt water) with a salt concentration of 2 grams per liter (gr/lit) flows into the tank at a constant rate of 3 liters per minute (lit/min) and the well-stirred mixture flows out of the tank at the same rate. Let $S(t)$ denote the mass (gr) of salt in the tank at time $t \geq 0$.

- (a) Write down an initial-value problem that governs $S(t)$.

Solution. Because water flows in and out of the tank at the same rate, the tank will contain 100 liters of salt water for every $t > 0$. The salt concentration of the water in the tank at time t will therefore be $S(t)/100$ gr/lit. Because this is also the concentration of the outflow, $S(t)$, the mass of salt in the tank at time t , will satisfy

$$\frac{dS}{dt} = \text{RATE IN} - \text{RATE OUT} = 2 \cdot 3 - \frac{S}{100} \cdot 3 = 6 - \frac{3}{100}S.$$

Because there is no salt in the tank initially, the initial-value problem that governs $S(t)$ is

$$\frac{dS}{dt} = 6 - \frac{3}{100}S, \quad S(0) = 0.$$

(b) Is $S(t)$ an increasing or decreasing function of t ? (Give your reasoning.)

Solution. We see from part (a) that

$$\frac{dS}{dt} = \frac{3}{100}(200 - S) > 0 \quad \text{for } S < 200,$$

whereby $S(t)$ is an increasing function of t that will approach the stationary value of 200 gr as $t \rightarrow \infty$.

(c) What is the behavior of $S(t)$ as $t \rightarrow \infty$? (Give your reasoning.)

Solution. The argument given for part (b) already shows that $S(t)$ is an increasing function of t that approaches the stationary value of 200 gr as $t \rightarrow \infty$.

(d) Derive an explicit formula for $S(t)$.

Solution. The differential equation given in the answer to part (a) is linear, so write it in the form

$$\frac{dS}{dt} + \frac{3}{100}S = 6.$$

An integrating factor is $e^{\frac{3}{100}t}$, whereby

$$\frac{d}{dt}(e^{\frac{3}{100}t}S) = 6e^{\frac{3}{100}t}.$$

This is the integrated to obtain

$$e^{\frac{3}{100}t}S = 200e^{\frac{3}{100}t} + c.$$

The integration constant c is found by setting $t = 0$ and $S = 0$, whereby

$$c = e^0 \cdot 0 - 200 \cdot e^0 = -200.$$

Then solving for S gives

$$S(t) = 200 - 200e^{-\frac{3}{100}t}.$$

(7) Suppose we are using the Runge-midpoint method to numerically approximate the solution of an initial-value problem over the time interval $[0, 5]$. By what factor would we expect the error to decrease when we increase the number of time steps taken from 500 to 2000?

Solution. The Runge-midpoint method is second order, which means its (global) error scales like h^2 where h is the step size. When the number of time steps taken increases from 500 to 2000, the step size h decreases by a factor of $1/4$. Therefore the error will decrease (like h^2) by a factor of $1/4^2 = 1/16$.

(8) Give an implicit general solution to each of the following differential equations.

(a) $\left(\frac{y}{x} + 3x\right) dx + (\log(x) - y) dy = 0.$

Solution. Because

$$\partial_y\left(\frac{y}{x} + 3x\right) = \frac{1}{x} = \partial_x(\log(x) - y) = \frac{1}{x},$$

the equation is *exact*. Therefore we can find $H(x, y)$ such that

$$\partial_x H(x, y) = \frac{y}{x} + 3x, \quad \partial_y H(x, y) = \log(x) - y.$$

The first of these equations implies that

$$H(x, y) = y \log(x) + \frac{3}{2}x^2 + h(y).$$

Plugging this into the second equation then shows that

$$\log(x) - y = \partial_y H(x, y) = \log(x) + h'(y).$$

Hence, $h'(y) = -y$, which yields $h(y) = -\frac{1}{2}y^2$. Therefore a general solution is governed implicitly by

$$y \log(x) + \frac{3}{2}x^2 - \frac{1}{2}y^2 = c, \quad \text{where } c \text{ is an arbitrary constant.}$$

(b) $(x^2 + y^3 + 2x) dx + 3y^2 dy = 0$.

Solution. Because

$$\partial_y(x^2 + y^3 + 2x) = 3y^2 \neq \partial_x(3y^2) = 0,$$

the equation is *not exact*. Seek an integrating factor $\mu(x, y)$ such that

$$\partial_y((x^2 + y^3 + 2x)\mu) = \partial_x(3y^2\mu).$$

This means that μ must satisfy

$$(x^2 + y^3 + 2x)\partial_y\mu + 3y^2\mu = 3y^2\partial_x\mu.$$

If we assume that μ depends only on x (so that $\partial_y\mu = 0$) then this reduces to

$$\mu = \partial_x\mu,$$

which depends only on x . We see from this that $\mu = e^x$ is an integrating factor. This implies that

$$(x^2 + y^3 + 2x)e^x dx + 3y^2e^x dy = 0 \quad \text{is exact.}$$

Therefore we can find $H(x, y)$ such that

$$\partial_x H(x, y) = (x^2 + y^3 + 2x)e^x, \quad \partial_y H(x, y) = 3y^2e^x.$$

The second of these equations implies that

$$H(x, y) = y^3e^x + h(x).$$

Plugging this into the first equation then yields

$$(x^2 + y^3 + 2x)e^x = \partial_x H(x, y) = y^3e^x + h'(x).$$

Hence, h satisfies

$$h'(x) = (x^2 + 2x)e^x.$$

This can be integrated to obtain $h(x) = x^2e^x$. Therefore a general solution is governed implicitly by

$$(y^3 + x^2)e^x = c, \quad \text{where } c \text{ is an arbitrary constant.}$$

- (9) A 2 kilogram (kg) mass initially at rest is dropped in a medium that offers a resistance of $v^2/40$ newtons ($= \text{kg m/sec}^2$) where v is the downward velocity (m/sec) of the mass. The gravitational acceleration is 9.8 m/sec^2 .

(a) What is the terminal velocity of the mass?

Solution. The terminal velocity is the velocity at which the force of resistance balances that of gravity. This happens when

$$\frac{1}{40}v^2 = mg = 2 \cdot 9.8.$$

Upon solving this for v one obtains

$$\begin{aligned} v &= \sqrt{40 \cdot 2 \cdot 9.8} \text{ m/sec} && \text{(full marks)} \\ &= \sqrt{4 \cdot 2 \cdot 98} = \sqrt{4 \cdot 2 \cdot 2 \cdot 49} \\ &= \sqrt{4^2 \cdot 7^2} = 4 \cdot 7 = 28 \text{ m/sec}. \end{aligned}$$

- (b) Write down an initial-value problem that governs v as a function of time. (You do not have to solve it!)

Solution. The net downward force on the falling mass is the force of gravity minus the force of resistance. By Newton ($ma = F$), this leads to

$$m \frac{dv}{dt} = mg - \frac{1}{40}v^2.$$

Because $m = 2$ and $g = 9.8$, and because the mass is initially at rest, this yields the initial-value problem

$$\frac{dv}{dt} = 9.8 - \frac{1}{80}v^2, \quad v(0) = 0.$$

- (10) Consider the following Matlab function m-file.

```
function [t,y] = solveit(tI, yI, tF, n)

t = zeros(n + 1, 1); y = zeros(n + 1, 1);
t(1) = tI; y(1) = yI; h = (tF - tI)/n;
for i = 1:n
z = t(i)^4 + y(i)^2;
t(i + 1) = t(i) + h;
y(i + 1) = y(i) + (h/2)*(z + t(i + 1)^4 + (y(i) + h*z)^2);
end
```

Suppose the input values are $tI = 1$, $yI = 1$, $tF = 5$, and $n = 20$.

- (a) What is the initial-value problem being approximated numerically?

Solution. The initial-value problem being approximated is

$$\frac{dy}{dt} = t^4 + y^2, \quad y(1) = 1.$$

Remark. You should not confuse the $y(1)$ above with the $y(1)$ appearing in the Matlab program. The first denotes the solution $y(t)$ of the initial-value problem evaluated at $t = 1$. The second denotes the first entry of the Matlab array y . Here they have the same value because $tI = 1$ but they will be different in general.

- (b) What is the numerical method being used?

Solution. The Runge-Trapezoidal method is being used.

- (c) What is the step size?

Solution. Because $tF = 5$, $tI = 1$, and $n = 20$, the step size is

$$h = \frac{tF - tI}{n} = \frac{5 - 1}{20} = \frac{4}{20} = .2.$$

Remark. You have to plug in the correct values for tF , tI , and n to get any credit.

- (d) What are the output values of $t(2)$ and $y(2)$?

Remark. Notice that this is asking for the values of the second entries of the Matlab arrays t and y produced by the above m-file. In particular, $y(2)$ is not the solution $y(t)$ of the initial-value problem evaluated at $t = 2$!

Solution. The step size is given by $h = .2$. The initial time and value are given by $t(1) = tI = 1$ and $y(1) = yI = 1$. By setting $i = 1$ inside the “for loop” we see that

$$\begin{aligned} z &= t(1)^4 + y(1)^2 = 1 + 1 = 2, \\ t(2) &= t(1) + h = 1 + .2 = 1.2, \\ y(2) &= y(1) + (h/2) (z + t(2)^4 + (y(1) + h z)^2) \\ &= 1 + .1(2 + (1.2)^4 + (1 + .2 \cdot 2)^2). \end{aligned}$$