Sample Problems for Third In-Class Exam
Math 246, Fall 2015, Professor David Levermore

(1) Compute the Laplace transform of $f(t) = t\,e^{3t}$ from its definition.

(2) Consider the following MATLAB commands.

```
>> syms t s Y; f = ['heaviside(t)*t^2 + heaviside(t − 3)*(3*t − t^2)'];
>> diffeqn = sym('D(D(y))(t) − 6*D(y)(t) + 10*y(t) = ' f);
>> eqntrans = laplace(diffeqn, t, s);
>> algeqn = subs(eqntrans, {'laplace(y(t),t,s),t,s)', 'y(0)', 'D(y)(0)'}, {Y, 2, 3});
>> ytrans = simplify(solve(algeqn, Y));
>> y = ilaplace(ytrans, s, t)
```

(a) Give the initial-value problem for $y(t)$ that is being solved.
(b) Find the Laplace transform $Y(s)$ of the solution $y(t)$.
You may refer to the table on the last page. DO NOT take the inverse Laplace transform to find $y(t)$, just solve for $Y(s)$!

(3) Find the Laplace transform $Y(s)$ of the solution $y(t)$ of the initial-value problem

$$y'' + 4y' + 13y = f(t)\,, \qquad y(0) = 4\,, \quad y'(0) = 1\,,$$

where

$$f(t) = \begin{cases} \cos(t) & \text{for } 0 \le t < 2\pi\,, \\ t - 2\pi & \text{for } t \ge 2\pi\,. \end{cases}$$

You may refer to the table on the last page. DO NOT take the inverse Laplace transform to find $y(t)$, just solve for $Y(s)$!

(4) Find the inverse Laplace transforms of the following functions. You may refer to the table on the last page.

(a) $F(s) = \dfrac{2}{(s+5)^2}\,,$

(b) $F(s) = \dfrac{3s}{s^2 - s - 6}\,,$

(c) $F(s) = \dfrac{(s-2)e^{-3s}}{s^2 - 4s + 5}\,.$

(5) Consider the matrices

$$\mathbf{A} = \begin{pmatrix} -i2 & 1+i \\ 2+i & -4 \end{pmatrix}\,, \qquad \mathbf{B} = \begin{pmatrix} 7 & 6 \\ 8 & 7 \end{pmatrix}\,.$$

Compute the matrices
(a) $\mathbf{A}^T$,
(b) $5\mathbf{A} - \mathbf{B}$,
(c) $\mathbf{AB}$,
(d) $\mathbf{B}^{-1}$.

(6) Transform the equation $u''' + t^2 u' - 3u = \sinh(2t)$ into a first-order system of ordinary differential equations.

(7) Consider two interconnected tanks filled with brine (salt water). The first tank contains 100 liters and the second contains 50 liters. Brine flows with a concentration of 2 grams of salt per liter flows into the first tank at a rate of 3 liters per hour. Well stirred brine flows from the first tank to the second at a rate of 5 liters per hour, from the second to the first at a rate of 2 liters per hour, and from the second into a drain at a rate of 3 liters per hour. At $t = 0$ there are 5 grams of salt in the first tank and 20 grams in the second. Give an initial-value problem that governs the amount of salt in each tank as a function of time.

(8) Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 3 & 3 \\ 4 & -1 \end{pmatrix} .$$

(a) Find all the eigenvalues of $\mathbf{A}$.
(b) For each eigenvalue of $\mathbf{A}$ find all of its eigenvectors.
(c) Diagonalize $\mathbf{A}$.

(9) Given that 1 is an eigenvalue of the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 1 & -1 \\ 0 & -1 & 3 \end{pmatrix} ,$$

find all the eigenvectors of $\mathbf{A}$ associated with 1.

(10) Consider the vector-valued functions $\mathbf{x}_1(t) = \begin{pmatrix} t^4 + 3 \\ 2t^2 \end{pmatrix}$, $\mathbf{x}_2(t) = \begin{pmatrix} t^2 \\ 3 \end{pmatrix}$.

(a) Compute the Wronskian $W[\mathbf{x}_1, \mathbf{x}_2](t)$.
(b) Find $\mathbf{A}(t)$ such that $\mathbf{x}_1$, $\mathbf{x}_2$ is a fundamental set of solutions to the system

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}(t)\mathbf{x} ,$$

wherever $W[\mathbf{x}_1, \mathbf{x}_2](t) \neq 0$.
(c) Give a fundamental matrix $\mathbf{\Psi}(t)$ for the system found in part (b).
(d) For the system found in part (b), solve the initial-value problem

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}(t)\mathbf{x} , \qquad \mathbf{x}(1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} .$$

(11) A 3×3 matrix $\mathbf{A}$ has the eigenpairs

$$\left(-3 , \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right) , \qquad \left(2 , \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right) , \qquad \left(5 , \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \right) .$$

(a) Give an invertible matrix $\mathbf{V}$ and a diagonal matrix $\mathbf{D}$ such that $e^{t\mathbf{A}} = \mathbf{V} e^{t\mathbf{D}} \mathbf{V}^{-1}$. (You do not have to compute either $\mathbf{V}^{-1}$ or $e^{t\mathbf{A}}$!)
(b) Give a fundamental matrix for the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

(12) Compute $e^{t\mathbf{A}}$ for the following matrices.

(a) $\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}$

(b) $\mathbf{A} = \begin{pmatrix} 6 & 4 \\ -1 & 2 \end{pmatrix}$

(13) Solve each of the following initial-value problems.

(a) $\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 5 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$

(b) $\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$

(14) Find a general solution for each of the following systems.

(a) $\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

(b) $\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

(c) $\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ -5 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

A Short Table of Laplace Transforms

$$\mathcal{L}[t^n e^{at}](s) = \frac{n!}{(s-a)^{n+1}} \qquad \text{for } s > a\,.$$

$$\mathcal{L}[e^{at}\cos(bt)](s) = \frac{s-a}{(s-a)^2+b^2} \qquad \text{for } s > a\,.$$

$$\mathcal{L}[e^{at}\sin(bt)](s) = \frac{b}{(s-a)^2+b^2} \qquad \text{for } s > a\,.$$

$$\mathcal{L}[t^n j(t)](s) = (-1)^n J^{(n)}(s) \qquad \text{where } J(s) = \mathcal{L}[j(t)](s)\,.$$

$$\mathcal{L}[e^{at} j(t)](s) = J(s-a) \qquad \text{where } J(s) = \mathcal{L}[j(t)](s)\,.$$

$$\mathcal{L}[u(t-c)j(t-c)](s) = e^{-cs}J(s) \qquad \text{where } J(s) = \mathcal{L}[j(t)](s)$$

and u is the unit step function.