

**Solutions of Sample Problems for Third In-Class Exam
Math 246, Fall 2015, Professor David Levermore**

(1) Compute the Laplace transform of $f(t) = t e^{3t}$ from its definition.

Solution. The definition of the Laplace transform gives

$$\mathcal{L}[f](s) = \lim_{T \rightarrow \infty} \int_0^T e^{-st} t e^{3t} dt = \lim_{T \rightarrow \infty} \int_0^T t e^{(3-s)t} dt.$$

This limit diverges to $+\infty$ for $s \leq 3$ because in that case

$$\int_0^T t e^{(3-s)t} dt \geq \int_0^T t dt = \frac{T^2}{2},$$

which clearly diverges to $+\infty$ as $T \rightarrow \infty$.

For $s > 3$ an integration by parts shows that

$$\begin{aligned} \int_0^T t e^{(3-s)t} dt &= t \frac{e^{(3-s)t}}{3-s} \Big|_0^T - \int_0^T \frac{e^{(3-s)t}}{3-s} dt \\ &= \left(t \frac{e^{(3-s)t}}{3-s} - \frac{e^{(3-s)t}}{(3-s)^2} \right) \Big|_0^T \\ &= \left(T \frac{e^{(3-s)T}}{3-s} - \frac{e^{(3-s)T}}{(3-s)^2} \right) + \frac{1}{(3-s)^2}. \end{aligned}$$

Hence, for $s > 3$ we have that

$$\begin{aligned} \mathcal{L}[f](s) &= \lim_{T \rightarrow \infty} \left[\left(T \frac{e^{(3-s)T}}{3-s} - \frac{e^{(3-s)T}}{(3-s)^2} \right) + \frac{1}{(3-s)^2} \right] \\ &= \frac{1}{(3-s)^2} + \lim_{T \rightarrow \infty} \left(T \frac{e^{(3-s)T}}{3-s} - \frac{e^{(3-s)T}}{(3-s)^2} \right) \\ &= \frac{1}{(3-s)^2}. \end{aligned}$$

(2) Consider the following MATLAB commands.

```
>> syms t s Y; f = ['heaviside(t)*t^2 + heaviside(t - 3)*(3*t - t^2)'];
>> diffeqn = sym('D(D(y))(t) - 6*D(y)(t) + 10*y(t) = ' f);
>> eqntrans = laplace(diffeqn, t, s);
>> algeqn = subs(eqntrans, {'laplace(y(t),t,s),t,s'}, 'y(0)', 'D(y)(0)'); {Y, 2, 3});
>> ytrans = simplify(solve(algeqn, Y));
>> y = ilaplace(ytrans, s, t)
```

(a) Give the initial-value problem for $y(t)$ that is being solved.

(b) Find the Laplace transform $Y(s)$ of the solution $y(t)$.

You may refer to the table on the last page. DO NOT take the inverse Laplace transform to find $y(t)$, just solve for $Y(s)$!

Solution (a). The initial-value problem for $y(t)$ that is being solved is

$$y'' - 6y' + 10y = f(t), \quad y(0) = 2, \quad y'(0) = 3,$$

where the forcing $f(t)$ can be expressed either as

$$f(t) = \begin{cases} t^2 & \text{for } 0 \leq t < 3, \\ 3t & \text{for } 3 \leq t, \end{cases}$$

or in terms of the unit step function as $f(t) = t^2 + u(t-3)(3t - t^2)$.

Solution (b). The Laplace transform of the initial-value problem is

$$\mathcal{L}[y''](s) - 6\mathcal{L}[y'](s) + 10\mathcal{L}[y](s) = \mathcal{L}[f](s),$$

where

$$\begin{aligned} \mathcal{L}[y](s) &= Y(s), \\ \mathcal{L}[y'](s) &= sY(s) - y(0) = sY(s) - 2, \\ \mathcal{L}[y''](s) &= s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - 2s - 3. \end{aligned}$$

To compute $\mathcal{L}[f](s)$, we first write $f(t)$ as

$$f(t) = t^2 + u(t-3)(3t - t^2) = t^2 + u(t-3)j(t-3),$$

where by setting $j(t-3) = 3t - t^2$ we see that

$$j(t) = 3(t+3) - (t+3)^2 = 3t + 9 - t^2 - 6t - 9 = -t^2 - 3t.$$

Referring to the table on the last page, item 1 with $a = 0$ and $n = 2$ and with $a = 0$ and $n = 1$ shows that

$$\mathcal{L}[t^2](s) = \frac{2}{s^3}, \quad \mathcal{L}[t](s) = \frac{1}{s^2},$$

whereby item 6 with $c = 3$ and $j(t) = -t^2 - 3t$ shows that

$$\mathcal{L}[u(t-3)j(t-3)](s) = e^{-3s}\mathcal{L}[j](s) = -e^{-3s}\mathcal{L}[t^2 + 3t](s) = -e^{-3s}\left(\frac{2}{s^3} + \frac{3}{s^2}\right).$$

Therefore

$$\mathcal{L}[f](s) = \mathcal{L}[t^2 + u(t-3)j(t-3)](s) = \frac{2}{s^3} - e^{-3s}\left(\frac{2}{s^3} + \frac{3}{s^2}\right).$$

The Laplace transform of the initial-value problem then becomes

$$(s^2Y(s) - 2s - 3) - 6(sY(s) - 2) + 10Y(s) = \frac{2}{s^3} - e^{-3s}\left(\frac{2}{s^3} + \frac{3}{s^2}\right),$$

which becomes

$$(s^2 - 6s + 10)Y(s) - 2s + 9 = \frac{2}{s^3} - e^{-3s}\left(\frac{2}{s^3} + \frac{3}{s^2}\right).$$

Therefore $Y(s)$ is given by

$$Y(s) = \frac{1}{s^2 - 6s + 10} \left(2s - 9 + \frac{2}{s^3} - e^{-3s} \left(\frac{2}{s^3} + \frac{3}{s^2} \right) \right).$$

(3) Find the Laplace transform $Y(s)$ of the solution $y(t)$ of the initial-value problem

$$y'' + 4y' + 13y = f(t), \quad y(0) = 4, \quad y'(0) = 1,$$

where

$$f(t) = \begin{cases} \cos(t) & \text{for } 0 \leq t < 2\pi, \\ t - 2\pi & \text{for } t \geq 2\pi. \end{cases}$$

You may refer to the table on the last page. DO NOT take the inverse Laplace transform to find $y(t)$, just solve for $Y(s)$!

Solution. The Laplace transform of the initial-value problem is

$$\mathcal{L}[y''](s) + 4\mathcal{L}[y'](s) + 13\mathcal{L}[y](s) = \mathcal{L}[f](s),$$

where

$$\mathcal{L}[y](s) = Y(s),$$

$$\mathcal{L}[y'](s) = sY(s) - y(0) = sY(s) - 4,$$

$$\mathcal{L}[y''](s) = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - 4s - 1.$$

To compute $\mathcal{L}[f](s)$, first write f as

$$\begin{aligned} f(t) &= (1 - u(t - 2\pi)) \cos(t) + u(t - 2\pi)(t - 2\pi) \\ &= \cos(t) - u(t - 2\pi) \cos(t) + u(t - 2\pi)(t - 2\pi) \\ &= \cos(t) - u(t - 2\pi) \cos(t - 2\pi) + u(t - 2\pi)(t - 2\pi). \end{aligned}$$

Referring to the table on the last page, item 6 with $c = 2\pi$ and $j(t) = \cos(t)$, item 6 with $c = 2\pi$ and $j(t) = t$, item 2 with $a = 0$ and $b = 1$, and item 1 with $n = 1$ and $a = 1$ then show that

$$\begin{aligned} \mathcal{L}[f](s) &= \mathcal{L}[\cos(t)](s) - \mathcal{L}[u(t - 2\pi) \cos(t - 2\pi)](s) + \mathcal{L}[u(t - 2\pi)(t - 2\pi)](s) \\ &= \mathcal{L}[\cos(t)](s) - e^{-2\pi s} \mathcal{L}[\cos(t)](s) + e^{-2\pi s} \mathcal{L}[t](s) \\ &= (1 - e^{-2\pi s}) \frac{s}{s^2 + 1} + e^{-2\pi s} \frac{1}{s^2}. \end{aligned}$$

The Laplace transform of the initial-value problem then becomes

$$(s^2Y(s) - 4s - 1) + 4(sY(s) - 4) + 13Y(s) = (1 - e^{-2\pi s}) \frac{s}{s^2 + 1} + e^{-2\pi s} \frac{1}{s^2},$$

which becomes

$$(s^2 + 4s + 13)Y(s) - 4s - 1 - 16 = (1 - e^{-2\pi s}) \frac{s}{s^2 + 1} + e^{-2\pi s} \frac{1}{s^2}.$$

Hence, $Y(s)$ is given by

$$Y(s) = \frac{1}{s^2 + 4s + 13} \left(4s + 17 + (1 - e^{-2\pi s}) \frac{s}{s^2 + 1} + e^{-2\pi s} \frac{1}{s^2} \right).$$

(4) Find the inverse Laplace transforms of the following functions. You may refer to the table on the last page.

$$(a) F(s) = \frac{2}{(s+5)^2},$$

Solution. Referring to the table on the last page, item 1 with $n = 1$ and $a = -5$ gives

$$\mathcal{L}[t e^{-5t}](s) = \frac{1}{(s+5)^2}.$$

Therefore we conclude that

$$\mathcal{L}^{-1}\left[\frac{2}{(s+5)^2}\right](t) = 2\mathcal{L}^{-1}\left[\frac{1}{(s+5)^2}\right](t) = 2t e^{-5t}.$$

$$(b) F(s) = \frac{3s}{s^2 - s - 6},$$

Solution. Because the denominator factors as $(s-3)(s+2)$, we have the partial fraction identity

$$\frac{3s}{s^2 - s - 6} = \frac{3s}{(s-3)(s+2)} = \frac{\frac{9}{5}}{s-3} + \frac{\frac{6}{5}}{s+2}.$$

Referring to the table on the last page, item 1 with $n = 0$ and $a = 3$, and with $n = 0$ and $a = -2$ gives

$$\mathcal{L}[e^{3t}](s) = \frac{1}{s-3}, \quad \mathcal{L}[e^{-2t}](s) = \frac{1}{s+2}.$$

Therefore we conclude that

$$\begin{aligned} \mathcal{L}^{-1}\left[\frac{3s}{s^2 - s - 6}\right](t) &= \mathcal{L}^{-1}\left[\frac{\frac{9}{5}}{s-3} + \frac{\frac{6}{5}}{s+2}\right](t) \\ &= \frac{9}{5}\mathcal{L}^{-1}\left[\frac{1}{s-3}\right](t) + \frac{6}{5}\mathcal{L}^{-1}\left[\frac{1}{s+2}\right](t) \\ &= \frac{9}{5}e^{3t} + \frac{6}{5}e^{-2t}. \end{aligned}$$

$$(c) F(s) = \frac{(s-2)e^{-3s}}{s^2 - 4s + 5}.$$

Solution. Complete the square in the denominator to get $(s-2)^2 + 1$. Referring to the table on the last page, item 2 with $a = 2$ and $b = 1$ gives

$$\mathcal{L}[e^{2t} \cos(t)](s) = \frac{s-2}{(s-2)^2 + 1}.$$

Item 6 with $c = 3$ and $j(t) = e^{2t} \cos(t)$ then gives

$$\mathcal{L}[u(t-3)e^{2(t-3)} \cos(t-3)](s) = e^{-3s} \frac{s-2}{(s-2)^2 + 1}.$$

Therefore we conclude that

$$\mathcal{L}^{-1}\left[e^{-3s} \frac{s-2}{s^2 - 4s + 5}\right](t) = u(t-3)e^{2(t-3)} \cos(t-3).$$

(5) Consider the matrices

$$\mathbf{A} = \begin{pmatrix} -i2 & 1+i \\ 2+i & -4 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 7 & 6 \\ 8 & 7 \end{pmatrix}.$$

Compute the matrices

(a) \mathbf{A}^T ,

Solution. The transpose of \mathbf{A} is

$$\mathbf{A}^T = \begin{pmatrix} -i2 & 2+i \\ 1+i & -4 \end{pmatrix}.$$

(b) $5\mathbf{A} - \mathbf{B}$,

Solution. The difference of $5\mathbf{A}$ and \mathbf{B} is given by

$$5\mathbf{A} - \mathbf{B} = \begin{pmatrix} -i10 & 5+i5 \\ 10+i5 & -20 \end{pmatrix} - \begin{pmatrix} 7 & 6 \\ 8 & 7 \end{pmatrix} = \begin{pmatrix} -7-i10 & -1+i5 \\ 2+i5 & -27 \end{pmatrix}.$$

(c) \mathbf{AB} ,

Solution. The product of \mathbf{A} and \mathbf{B} is given by

$$\begin{aligned} \mathbf{AB} &= \begin{pmatrix} -i2 & 1+i \\ 2+i & -4 \end{pmatrix} \begin{pmatrix} 7 & 6 \\ 8 & 7 \end{pmatrix} \\ &= \begin{pmatrix} -i2 \cdot 7 + (1+i) \cdot 8 & -i2 \cdot 6 + (1+i) \cdot 7 \\ (2+i) \cdot 7 - 4 \cdot 8 & (2+i) \cdot 6 - 4 \cdot 7 \end{pmatrix} \\ &= \begin{pmatrix} 8-i6 & 7-i5 \\ -18+i7 & -16+i6 \end{pmatrix}. \end{aligned}$$

(d) \mathbf{B}^{-1} .

Solution. Observe that it is clear that \mathbf{B} has an inverse because

$$\det(\mathbf{B}) = \det \begin{pmatrix} 7 & 6 \\ 8 & 7 \end{pmatrix} = 7 \cdot 7 - 6 \cdot 8 = 49 - 48 = 1.$$

The inverse of \mathbf{B} is given by

$$\mathbf{B}^{-1} = \frac{1}{\det(\mathbf{B})} \begin{pmatrix} 7 & -6 \\ -8 & 7 \end{pmatrix} = \begin{pmatrix} 7 & -6 \\ -8 & 7 \end{pmatrix}.$$

(6) Transform the equation $u''' + t^2u' - 3u = \sinh(2t)$ into a first-order system of ordinary differential equations.

Solution. Because the equation is third order, the first order system must have dimension three. The simplest such first order system is

$$\frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ \sinh(2t) + 3x_1 - t^2x_2 \end{pmatrix}, \quad \text{where} \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} u \\ u' \\ u'' \end{pmatrix}.$$

- (7) Consider two interconnected tanks filled with brine (salt water). The first tank contains 100 liters and the second contains 50 liters. Brine flows with a concentration of 2 grams of salt per liter flows into the first tank at a rate of 3 liters per hour. Well stirred brine flows from the first tank to the second at a rate of 5 liters per hour, from the second to the first at a rate of 2 liters per hour, and from the second into a drain at a rate of 3 liters per hour. At $t = 0$ there are 5 grams of salt in the first tank and 20 grams in the second. Give an initial-value problem that governs the amount of salt in each tank as a function of time.

Solution. The rates work out so there will always be 100 liters of brine in the first tank and 50 liters in the second. Let $S_1(t)$ be the grams of salt in the first tank and $S_2(t)$ be the grams of salt in the second tank. These are governed by the initial-value problem

$$\begin{aligned}\frac{dS_1}{dt} &= 2 \cdot 3 + \frac{S_2}{50} 2 - \frac{S_1}{100} 5, & S_1(0) &= 5, \\ \frac{dS_2}{dt} &= \frac{S_1}{100} 5 - \frac{S_2}{50} 2 - \frac{S_2}{50} 3, & S_2(0) &= 20.\end{aligned}$$

You could leave the answer in the above form. It can however be simplified to

$$\begin{aligned}\frac{dS_1}{dt} &= 6 + \frac{S_2}{25} - \frac{S_1}{20}, & S_1(0) &= 5, \\ \frac{dS_2}{dt} &= \frac{S_1}{20} - \frac{S_2}{10}, & S_2(0) &= 20.\end{aligned}$$

- (8) Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 3 & 3 \\ 4 & -1 \end{pmatrix}.$$

- (a) Find all the eigenvalues of \mathbf{A} .

Solution. The characteristic polynomial of \mathbf{A} is given by

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 2z - 15 = (z - 1)^2 - 16.$$

The eigenvalues of \mathbf{A} are the roots of this polynomial, which are 1 ± 4 , or simply -3 and 5 .

- (b) For each eigenvalue of \mathbf{A} find all of its eigenvectors.

Solution (using the Cayley-Hamilton method from notes). We have

$$\mathbf{A} + 3\mathbf{I} = \begin{pmatrix} 6 & 3 \\ 4 & 2 \end{pmatrix}, \quad \mathbf{A} - 5\mathbf{I} = \begin{pmatrix} -2 & 3 \\ 4 & -6 \end{pmatrix}.$$

Every nonzero column of $\mathbf{A} - 5\mathbf{I}$ has the form

$$\alpha_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad \text{for some } \alpha_1 \neq 0.$$

These are all the eigenvectors associated with -3 . Similarly, every nonzero column of $\mathbf{A} + 3\mathbf{I}$ has the form

$$\alpha_2 \begin{pmatrix} 3 \\ 2 \end{pmatrix} \quad \text{for some } \alpha_2 \neq 0.$$

These are all the eigenvectors associated with 5 .

(c) Diagonalize \mathbf{A} .

Solution. If we use the eigenpairs

$$\left(-3, \begin{pmatrix} 1 \\ -2 \end{pmatrix}\right), \quad \left(5, \begin{pmatrix} 3 \\ 2 \end{pmatrix}\right),$$

then set

$$\mathbf{V} = \begin{pmatrix} 1 & 3 \\ -2 & 2 \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} -3 & 0 \\ 0 & 5 \end{pmatrix}.$$

Because $\det(\mathbf{V}) = 1 \cdot 2 - (-2) \cdot 3 = 2 + 6 = 8$, we see that

$$\mathbf{V}^{-1} = \frac{1}{8} \begin{pmatrix} 2 & -3 \\ 2 & 1 \end{pmatrix}.$$

We conclude that \mathbf{A} has the diagonalization

$$\mathbf{A} = \mathbf{VDV}^{-1} = \begin{pmatrix} 1 & 3 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} -3 & 0 \\ 0 & 5 \end{pmatrix} \frac{1}{8} \begin{pmatrix} 2 & -3 \\ 2 & 1 \end{pmatrix}.$$

You do not have to multiply these matrices out. Had we started with different eigenpairs, the steps would be the same as above but we would obtain a different diagonalization.

(9) Given that 1 is an eigenvalue of the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 1 & -1 \\ 0 & -1 & 3 \end{pmatrix},$$

find all the eigenvectors of \mathbf{A} associated with 1.

Solution. The eigenvectors of \mathbf{A} associated with 1 are all nonzero vectors \mathbf{v} that satisfy $\mathbf{A}\mathbf{v} = \mathbf{v}$. Equivalently, they are all nonzero vectors \mathbf{v} that satisfy $(\mathbf{A} - \mathbf{I})\mathbf{v} = \mathbf{0}$, which is

$$\begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \mathbf{0}.$$

The entries of \mathbf{v} thereby satisfy the homogeneous linear algebraic system

$$\begin{aligned} v_1 - v_2 + v_3 &= 0, \\ v_1 - v_3 &= 0, \\ -v_2 + 2v_3 &= 0. \end{aligned}$$

We may solve this system either by elimination or by row reduction. By either method we find that its general solution is

$$v_1 = \alpha, \quad v_2 = 2\alpha, \quad v_3 = \alpha, \quad \text{for any constant } \alpha.$$

The eigenvectors of \mathbf{A} associated with 1 therefore have the form

$$\alpha \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \quad \text{for any nonzero constant } \alpha.$$

(10) Consider the vector-valued functions $\mathbf{x}_1(t) = \begin{pmatrix} t^4 + 3 \\ 2t^2 \end{pmatrix}$, $\mathbf{x}_2(t) = \begin{pmatrix} t^2 \\ 3 \end{pmatrix}$.

(a) Compute the Wronskian $W[\mathbf{x}_1, \mathbf{x}_2](t)$.

Solution.

$$W[\mathbf{x}_1, \mathbf{x}_2](t) = \det \begin{pmatrix} t^4 + 3 & t^2 \\ 2t^2 & 3 \end{pmatrix} = 3t^4 + 9 - 2t^4 = t^4 + 9.$$

(b) Find $\mathbf{A}(t)$ such that $\mathbf{x}_1, \mathbf{x}_2$ is a fundamental set of solutions to $\frac{d\mathbf{x}}{dt} = \mathbf{A}(t)\mathbf{x}$ wherever $W[\mathbf{x}_1, \mathbf{x}_2](t) \neq 0$.

Solution. Let $\Psi(t) = \begin{pmatrix} t^4 + 3 & t^2 \\ 2t^2 & 3 \end{pmatrix}$. Because $\frac{d\Psi(t)}{dt} = \mathbf{A}(t)\Psi(t)$, we have

$$\begin{aligned} \mathbf{A}(t) &= \frac{d\Psi(t)}{dt} \Psi(t)^{-1} = \begin{pmatrix} 4t^3 & 2t \\ 4t & 0 \end{pmatrix} \begin{pmatrix} t^4 + 3 & t^2 \\ 2t^2 & 3 \end{pmatrix}^{-1} \\ &= \frac{1}{t^4 + 9} \begin{pmatrix} 4t^3 & 2t \\ 4t & 0 \end{pmatrix} \begin{pmatrix} 3 & -t^2 \\ -2t^2 & t^4 + 3 \end{pmatrix} = \frac{1}{t^4 + 9} \begin{pmatrix} 8t^3 & 6t - 2t^5 \\ 12t & -4t^3 \end{pmatrix}. \end{aligned}$$

(c) Give a fundamental matrix $\Psi(t)$ for the system found in part (b).

Solution. Because $\mathbf{x}_1(t), \mathbf{x}_2(t)$ is a fundamental set of solutions to the system found in part (b), a fundamental matrix for the system found in part (b) is simply given by

$$\Psi(t) = (\mathbf{x}_1(t) \quad \mathbf{x}_2(t)) = \begin{pmatrix} t^4 + 3 & t^2 \\ 2t^2 & 3 \end{pmatrix}.$$

(d) For the system found in part (b), solve the initial-value problem

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}(t)\mathbf{x}, \quad \mathbf{x}(1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Solution. Because a fundamental matrix $\Psi(t)$ for the system found in part (b) was given in part (c), the solution of the initial-value problem is

$$\begin{aligned} \mathbf{x}(t) &= \Psi(t)\Psi(1)^{-1}\mathbf{x}(1) = \begin{pmatrix} t^4 + 3 & t^2 \\ 2t^2 & 3 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} t^4 + 3 & t^2 \\ 2t^2 & 3 \end{pmatrix} \frac{1}{10} \begin{pmatrix} 3 & -1 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \frac{1}{10} \begin{pmatrix} t^4 + 3 & t^2 \\ 2t^2 & 3 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \frac{1}{10} \begin{pmatrix} 3t^4 + 9 - 2t^2 \\ 6t^2 - 6 \end{pmatrix}. \end{aligned}$$

Alternative Solution. Because $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$ is a fundamental set of solutions to the system found in part (b), a general solution is given by

$$\mathbf{x}(t) = c_1 \mathbf{x}_1(t) + c_2 \mathbf{x}_2(t) = c_1 \begin{pmatrix} t^4 + 3 \\ 2t^2 \end{pmatrix} + c_2 \begin{pmatrix} t^2 \\ 3 \end{pmatrix}.$$

The initial condition then implies that

$$\mathbf{x}(1) = c_1 \begin{pmatrix} 4 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 4c_1 + c_2 \\ 2c_1 + 3c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

from which we see that $c_1 = \frac{3}{10}$ and $c_2 = -\frac{1}{5}$. The solution of the initial-value problem is thereby

$$\mathbf{x}(t) = \frac{3}{10} \begin{pmatrix} t^4 + 3 \\ 2t^2 \end{pmatrix} - \frac{1}{5} \begin{pmatrix} t^2 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{3}{10}t^4 - \frac{1}{5}t^2 + \frac{9}{10} \\ \frac{3}{5}t^2 - \frac{3}{5} \end{pmatrix}.$$

(11) A 3×3 matrix \mathbf{A} has the eigenpairs

$$\left(-3, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right), \quad \left(2, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}\right), \quad \left(5, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}\right).$$

- (a) Give an invertible matrix \mathbf{V} and a diagonal matrix \mathbf{D} such that $e^{t\mathbf{A}} = \mathbf{V}e^{t\mathbf{D}}\mathbf{V}^{-1}$.
 (You do not have to compute either \mathbf{V}^{-1} or $e^{t\mathbf{A}}$!)
 (b) Give a fundamental matrix for the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

Solution (a). One choice for \mathbf{V} and \mathbf{D} is

$$\mathbf{V} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 0 & 1 & 2 \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} -3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}.$$

Solution (b). Use the given eigenpairs to construct the special solutions

$$\mathbf{x}_1(t) = e^{-3t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{x}_2(t) = e^{2t} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{x}_3(t) = e^{5t} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix},$$

Then a fundamental matrix for the system is

$$\mathbf{\Psi}(t) = (\mathbf{x}_1(t) \quad \mathbf{x}_2(t) \quad \mathbf{x}_3(t)) = \begin{pmatrix} e^{-3t} & -e^{2t} & e^{5t} \\ e^{-3t} & e^{2t} & -e^{5t} \\ 0 & e^{2t} & 2e^{5t} \end{pmatrix}.$$

Alternative Solution (b). Given the \mathbf{V} and \mathbf{D} from part (a), a fundamental matrix for the system is

$$\mathbf{\Psi}(t) = \mathbf{V}e^{t\mathbf{D}} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} e^{-3t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{5t} \end{pmatrix} = \begin{pmatrix} e^{-3t} & -e^{2t} & e^{5t} \\ e^{-3t} & e^{2t} & -e^{5t} \\ 0 & e^{2t} & 2e^{5t} \end{pmatrix}.$$

(12) Compute $e^{t\mathbf{A}}$ for the following matrices.

(a) $\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}$

Solution. The characteristic polynomial of \mathbf{A} is given by

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 2z - 3 = (z - 1)^2 - 4.$$

The eigenvalues of \mathbf{A} are the roots of this polynomial, which are 1 ± 2 . Hence,

$$\begin{aligned} e^{t\mathbf{A}} &= e^t \left[\cosh(2t)\mathbf{I} + \frac{\sinh(2t)}{2}(\mathbf{A} - \mathbf{I}) \right] \\ &= e^t \left[\cosh(2t) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\sinh(2t)}{2} \begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix} \right] \\ &= e^t \begin{pmatrix} \cosh(2t) & 2\sinh(2t) \\ \frac{1}{2}\sinh(2t) & \cosh(2t) \end{pmatrix}. \end{aligned}$$

Alternative Solution. The characteristic polynomial of \mathbf{A} is

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 2z - 3 = (z + 1)(z - 3).$$

The associated second-order general initial-value problem is

$$y'' - 2y' - 3y = 0, \quad y(0) = y_0, \quad y'(0) = y_1.$$

This has the general solution $y(t) = c_1e^{3t} + c_2e^{-t}$. Because $y'(t) = 3c_1e^{3t} - c_2e^{-t}$, the general initial conditions yield

$$y_0 = y(0) = c_1 + c_2, \quad y_1 = y'(0) = 3c_1 - c_2.$$

This system can be solved to obtain

$$c_1 = \frac{y_0 + y_1}{4}, \quad c_2 = \frac{3y_0 - y_1}{4}.$$

The solution of the general initial-value problem is thereby

$$y(t) = \frac{y_0 + y_1}{4} e^{3t} + \frac{3y_0 - y_1}{4} e^{-t} = \frac{e^{3t} + 3e^{-t}}{4} y_0 + \frac{e^{3t} - e^{-t}}{4} y_1.$$

Therefore the associated natural fundamental set of solutions is

$$N_0(t) = \frac{e^{3t} + 3e^{-t}}{4}, \quad N_1(t) = \frac{e^{3t} - e^{-t}}{4},$$

whereby

$$\begin{aligned} e^{t\mathbf{A}} &= N_0(t)\mathbf{I} + N_1(t)\mathbf{A} = \frac{e^{3t} + 3e^{-t}}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{e^{3t} - e^{-t}}{4} \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} \\ &= \frac{1}{4} \begin{pmatrix} 2e^{3t} + 2e^{-t} & 4e^{3t} - 4e^{-t} \\ e^{3t} - e^{-t} & 2e^{3t} + 2e^{-t} \end{pmatrix}. \end{aligned}$$

Second Alternative Solution. The characteristic polynomial of \mathbf{A} is

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 2z - 3 = (z + 1)(z - 3).$$

The eigenvalues of \mathbf{A} are the roots of this polynomial, which are -1 and 3 . Because

$$\mathbf{A} + \mathbf{I} = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}, \quad \mathbf{A} - 3\mathbf{I} = \begin{pmatrix} -2 & 4 \\ 1 & -2 \end{pmatrix},$$

we can read off that \mathbf{A} has the eigenpairs

$$\left(-1, \begin{pmatrix} 2 \\ -1 \end{pmatrix}\right), \quad \left(3, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right).$$

Set

$$\mathbf{V} = \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}.$$

Because $\det(\mathbf{V}) = 4$, we see that

$$\begin{aligned} e^{t\mathbf{A}} &= \mathbf{V}e^{t\mathbf{D}}\mathbf{V}^{-1} = \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{3t} \end{pmatrix} \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{3t} \end{pmatrix} \frac{1}{4} \begin{pmatrix} 1 & -2 \\ 1 & 2 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} e^{-t} & -2e^{-t} \\ e^{3t} & 2e^{3t} \end{pmatrix} \\ &= \frac{1}{4} \begin{pmatrix} 2e^{-t} + 2e^{3t} & 4e^{3t} - 4e^{-t} \\ e^{3t} - e^{-t} & 2e^{-t} + 2e^{3t} \end{pmatrix}. \end{aligned}$$

(b) $\mathbf{A} = \begin{pmatrix} 6 & 4 \\ -1 & 2 \end{pmatrix}$

Solution. The characteristic polynomial of \mathbf{A} is given by

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 8z + 16 = (z - 4)^2.$$

The eigenvalues of \mathbf{A} are the roots of this polynomial, which is 4 , a double root. Hence,

$$\begin{aligned} e^{t\mathbf{A}} &= e^{4t} \left[\mathbf{I} + t(\mathbf{A} - 4\mathbf{I}) \right] = e^{4t} \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t \begin{pmatrix} 2 & 4 \\ -1 & -2 \end{pmatrix} \right] \\ &= e^{4t} \begin{pmatrix} 1 + 2t & 4t \\ -t & 1 - 2t \end{pmatrix}. \end{aligned}$$

Alternative Solution. The characteristic polynomial of \mathbf{A} is

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 8z + 16 = (z - 4)^4.$$

The associated second-order general initial-value problem is

$$y'' - 8y' + 16y = 0, \quad y(0) = y_0, \quad y'(0) = y_1.$$

This has the general solution $y(t) = c_1 e^{4t} + c_2 t e^{4t}$. Because $y'(t) = 4c_1 e^{3t} + 4c_2 t e^{-t} + c_2 e^{4t}$, the general initial conditions yield

$$y_0 = y(0) = c_1, \quad y_1 = y'(0) = 4c_1 + c_2.$$

This system can be solved to obtain $c_1 = y_0$ and $c_2 = y_1 - 4y_0$. The solution of the general initial-value problem is thereby

$$y(t) = y_0 e^{4t} + (y_1 - 4y_0)t e^{4t} = (1 - 4t)e^{4t} y_0 + t e^{4t} y_1.$$

Therefore the associated natural fundamental set of solutions is

$$N_0(t) = (1 - 4t)e^{4t}, \quad N_1(t) = te^{4t},$$

whereby

$$\begin{aligned} e^{t\mathbf{A}} &= N_0(t)\mathbf{I} + N_1(t)\mathbf{A} = (1 - 4t)e^{4t} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + te^{4t} \begin{pmatrix} 6 & 4 \\ -1 & 2 \end{pmatrix} \\ &= e^{4t} \begin{pmatrix} 1 + 2t & 4t \\ -t & 1 - 2t \end{pmatrix}. \end{aligned}$$

(13) Solve each of the following initial-value problems.

$$(a) \quad \frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 5 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Solution. The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} 2 & 2 \\ 5 & -1 \end{pmatrix}$ is given by

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - z - 12 = (z + 3)(z - 4).$$

The eigenvalues of \mathbf{A} are the roots of this polynomial, which are -3 and 4 . These have the form $\frac{1}{2} \pm \frac{7}{2}$, whereby

$$\begin{aligned} e^{t\mathbf{A}} &= e^{\frac{1}{2}t} \left[\cosh\left(\frac{7}{2}t\right)\mathbf{I} + \frac{\sinh\left(\frac{7}{2}t\right)}{\frac{7}{2}}(\mathbf{A} - \frac{1}{2}\mathbf{I}) \right] \\ &= e^{\frac{1}{2}t} \left[\cosh\left(\frac{7}{2}t\right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\sinh\left(\frac{7}{2}t\right)}{\frac{7}{2}} \begin{pmatrix} \frac{3}{2} & 2 \\ 5 & -\frac{3}{2} \end{pmatrix} \right] \\ &= e^{\frac{1}{2}t} \begin{pmatrix} \cosh\left(\frac{7}{2}t\right) + \frac{3}{7}\sinh\left(\frac{7}{2}t\right) & \frac{4}{7}\sinh\left(\frac{7}{2}t\right) \\ \frac{10}{7}\sinh\left(\frac{7}{2}t\right) & \cosh\left(\frac{7}{2}t\right) - \frac{3}{7}\sinh\left(\frac{7}{2}t\right) \end{pmatrix}. \end{aligned}$$

Therefore the solution of the initial-value problem is

$$\begin{aligned} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= e^{t\mathbf{A}} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = e^{t\mathbf{A}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &= e^{\frac{1}{2}t} \begin{pmatrix} \cosh\left(\frac{7}{2}t\right) + \frac{3}{7}\sinh\left(\frac{7}{2}t\right) & \frac{4}{7}\sinh\left(\frac{7}{2}t\right) \\ \frac{10}{7}\sinh\left(\frac{7}{2}t\right) & \cosh\left(\frac{7}{2}t\right) - \frac{3}{7}\sinh\left(\frac{7}{2}t\right) \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &= e^{\frac{1}{2}t} \begin{pmatrix} \cosh\left(\frac{7}{2}t\right) - \frac{1}{7}\sinh\left(\frac{7}{2}t\right) \\ -\cosh\left(\frac{7}{2}t\right) + \frac{13}{7}\sinh\left(\frac{7}{2}t\right) \end{pmatrix}. \end{aligned}$$

$$(b) \quad \frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Solution. The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix}$ is given by

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 2z + 5 = (z - 1)^2 + 4.$$

The eigenvalues of \mathbf{A} are the roots of this polynomial, which are $1 \pm i2$. Hence,

$$\begin{aligned} e^{t\mathbf{A}} &= e^t \left[\cos(2t)\mathbf{I} + \frac{\sin(2t)}{2}(\mathbf{A} - \mathbf{I}) \right] \\ &= e^t \left[\cos(2t) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\sin(2t)}{2} \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} \right] \\ &= e^t \begin{pmatrix} \cos(2t) & \frac{1}{2} \sin(2t) \\ -2 \sin(2t) & \cos(2t) \end{pmatrix}. \end{aligned}$$

Therefore the solution of the initial-value problem is

$$\begin{aligned} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= e^{t\mathbf{A}} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = e^{t\mathbf{A}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= e^t \begin{pmatrix} \cos(2t) & \frac{1}{2} \sin(2t) \\ -2 \sin(2t) & \cos(2t) \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= e^t \begin{pmatrix} \cos(2t) + \frac{1}{2} \sin(2t) \\ -2 \sin(2t) + \cos(2t) \end{pmatrix}. \end{aligned}$$

Remark. We could have used other methods to compute $e^{t\mathbf{A}}$ in each part of the above problem. Alternatively, we could have constructed a fundamental matrix $\Psi(t)$ and then determined \mathbf{c} so that $\Psi(t)\mathbf{c}$ satisfies the initial conditions.

(14) Find a general solution for each of the following systems.

$$(a) \quad \frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Solution. The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$ is given by

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 2z + 1 = (z - 1)^2.$$

The eigenvalues of \mathbf{A} are the roots of this polynomial, which is 1, a double root. Hence,

$$\begin{aligned} e^{t\mathbf{A}} &= e^t [\mathbf{I} + t(\mathbf{A} - \mathbf{I})] = e^t \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} \right] \\ &= e^t \begin{pmatrix} 1 + 2t & -4t \\ t & 1 - 2t \end{pmatrix}. \end{aligned}$$

Therefore a general solution is

$$\begin{aligned} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= e^{t\mathbf{A}} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = e^t \begin{pmatrix} 1 + 2t & -4t \\ t & 1 - 2t \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &= c_1 e^t \begin{pmatrix} 1 + 2t \\ t \end{pmatrix} + c_2 e^t \begin{pmatrix} -4t \\ 1 - 2t \end{pmatrix}. \end{aligned}$$

$$(b) \frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Solution. The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix}$ is given by

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 + 16.$$

The eigenvalues of \mathbf{A} are the roots of this polynomial, which are $\pm i4$. Hence,

$$\begin{aligned} e^{t\mathbf{A}} &= \left[\cos(4t)\mathbf{I} + \frac{\sin(4t)}{4}\mathbf{A} \right] = \left[\cos(4t) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\sin(4t)}{4} \begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix} \right] \\ &= \begin{pmatrix} \cos(4t) + \frac{1}{2}\sin(4t) & -\frac{5}{4}\sin(4t) \\ \sin(4t) & \cos(4t) - \frac{1}{2}\sin(4t) \end{pmatrix}. \end{aligned}$$

Therefore a general solution is

$$\begin{aligned} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= e^{t\mathbf{A}} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \cos(4t) + \frac{1}{2}\sin(4t) & -\frac{5}{4}\sin(4t) \\ \sin(4t) & \cos(4t) - \frac{1}{2}\sin(4t) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &= c_1 \begin{pmatrix} \cos(4t) + \frac{1}{2}\sin(4t) \\ \sin(4t) \end{pmatrix} + c_2 \begin{pmatrix} -\frac{5}{4}\sin(4t) \\ \cos(4t) - \frac{1}{2}\sin(4t) \end{pmatrix}. \end{aligned}$$

Alternative Solution. The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix}$ is

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 + 16.$$

The eigenvalues of \mathbf{A} are the roots of this polynomial, which are $\pm i4$. Because

$$\mathbf{A} - i4\mathbf{I} = \begin{pmatrix} 2 - i4 & -5 \\ 4 & -2 - i4 \end{pmatrix}, \quad \mathbf{A} + i4\mathbf{I} = \begin{pmatrix} 2 + i4 & -5 \\ 4 & -2 + i4 \end{pmatrix},$$

we can read off that \mathbf{A} has the eigenpairs

$$\left(i4, \begin{pmatrix} 1 + i2 \\ 2 \end{pmatrix} \right), \quad \left(-i4, \begin{pmatrix} 1 - i2 \\ 2 \end{pmatrix} \right).$$

Therefore the system has the complex-valued solution

$$\begin{aligned} e^{i4t} \begin{pmatrix} 1 + i2 \\ 2 \end{pmatrix} &= (\cos(4t) + i\sin(4t)) \begin{pmatrix} 1 + i2 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} \cos(4t) - 2\sin(4t) + i2\cos(4t) + i\sin(4t) \\ 2\cos(4t) + i2\sin(4t) \end{pmatrix}. \end{aligned}$$

By taking real and imaginary parts, we obtain the two real solutions

$$\begin{aligned} \mathbf{x}_1(t) &= \operatorname{Re} \left(e^{i4t} \begin{pmatrix} 1 + i2 \\ 2 \end{pmatrix} \right) = \begin{pmatrix} \cos(4t) - 2\sin(4t) \\ 2\cos(4t) \end{pmatrix}, \\ \mathbf{x}_2(t) &= \operatorname{Im} \left(e^{i4t} \begin{pmatrix} 1 + i2 \\ 2 \end{pmatrix} \right) = \begin{pmatrix} 2\cos(4t) + \sin(4t) \\ 2\sin(4t) \end{pmatrix}. \end{aligned}$$

Therefore a general solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \begin{pmatrix} \cos(4t) - 2\sin(4t) \\ 2\cos(4t) \end{pmatrix} + c_2 \begin{pmatrix} 2\cos(4t) + \sin(4t) \\ 2\sin(4t) \end{pmatrix}.$$

$$(c) \frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ -5 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Solution. The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} 5 & 4 \\ -5 & 1 \end{pmatrix}$ is given by

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 6z + 25 = (z - 3)^2 + 16.$$

The eigenvalues of \mathbf{A} are the roots of this polynomial, which are $3 \pm i4$. Hence,

$$\begin{aligned} e^{t\mathbf{A}} &= e^{3t} \left[\cos(4t)\mathbf{I} + \frac{\sin(4t)}{4}(\mathbf{A} - 3\mathbf{I}) \right] \\ &= e^{3t} \left[\cos(4t) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\sin(4t)}{4} \begin{pmatrix} 2 & 4 \\ -5 & -2 \end{pmatrix} \right] \\ &= e^{3t} \begin{pmatrix} \cos(4t) + \frac{1}{2}\sin(4t) & \sin(4t) \\ -\frac{5}{4}\sin(4t) & \cos(4t) - \frac{1}{2}\sin(4t) \end{pmatrix}. \end{aligned}$$

Therefore a general solution is

$$\begin{aligned} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= e^{t\mathbf{A}} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = e^{3t} \begin{pmatrix} \cos(4t) + \frac{1}{2}\sin(4t) & \sin(4t) \\ -\frac{5}{4}\sin(4t) & \cos(4t) - \frac{1}{2}\sin(4t) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &= c_1 e^{3t} \begin{pmatrix} \cos(4t) + \frac{1}{2}\sin(4t) \\ -\frac{5}{4}\sin(4t) \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} \sin(4t) \\ \cos(4t) - \frac{1}{2}\sin(4t) \end{pmatrix}. \end{aligned}$$

Alternative Solution. The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} 5 & 4 \\ -5 & 1 \end{pmatrix}$ is

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 6z + 25 = (z - 3)^2 + 16.$$

The eigenvalues of \mathbf{A} are the roots of this polynomial, which are $3 \pm i4$. Because

$$\mathbf{A} - (3 + i4)\mathbf{I} = \begin{pmatrix} 2 - i4 & 4 \\ -5 & -2 - i4 \end{pmatrix}, \quad \mathbf{A} - (3 - i4)\mathbf{I} = \begin{pmatrix} 2 + i4 & 4 \\ -5 & -2 + i4 \end{pmatrix},$$

we can read off that \mathbf{A} has the eigenpairs

$$\left(3 + i4, \begin{pmatrix} -2 \\ 1 - i2 \end{pmatrix} \right), \quad \left(3 - i4, \begin{pmatrix} -2 \\ 1 + i2 \end{pmatrix} \right).$$

Therefore the system has the complex-valued solution

$$\begin{aligned} e^{(3+i4)t} \begin{pmatrix} -2 \\ 1 - i2 \end{pmatrix} &= e^{3t} (\cos(4t) + i\sin(4t)) \begin{pmatrix} -2 \\ 1 - i2 \end{pmatrix} \\ &= e^{3t} \begin{pmatrix} -2\cos(4t) - i2\sin(4t) \\ \cos(4t) + 2\sin(4t) + i\sin(4t) - i2\cos(4t) \end{pmatrix}. \end{aligned}$$

By taking real and imaginary parts, we obtain the two real solutions

$$\begin{aligned} \mathbf{x}_1(t) &= \operatorname{Re} \left(e^{(3+i4)t} \begin{pmatrix} -2 \\ 1 - i2 \end{pmatrix} \right) = e^{3t} \begin{pmatrix} -2\cos(4t) \\ \cos(4t) + 2\sin(4t) \end{pmatrix}, \\ \mathbf{x}_2(t) &= \operatorname{Im} \left(e^{(3+i4)t} \begin{pmatrix} -2 \\ 1 - i2 \end{pmatrix} \right) = e^{3t} \begin{pmatrix} -2\sin(4t) \\ \sin(4t) - 2\cos(4t) \end{pmatrix}. \end{aligned}$$

Therefore a general solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{3t} \begin{pmatrix} -2 \cos(4t) \\ \cos(4t) + 2 \sin(4t) \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} -2 \sin(4t) \\ \sin(4t) - 2 \cos(4t) \end{pmatrix}.$$

A Short Table of Laplace Transforms

$$\mathcal{L}[t^n e^{at}](s) = \frac{n!}{(s-a)^{n+1}} \quad \text{for } s > a.$$

$$\mathcal{L}[e^{at} \cos(bt)](s) = \frac{s-a}{(s-a)^2 + b^2} \quad \text{for } s > a.$$

$$\mathcal{L}[e^{at} \sin(bt)](s) = \frac{b}{(s-a)^2 + b^2} \quad \text{for } s > a.$$

$$\mathcal{L}[t^n j(t)](s) = (-1)^n J^{(n)}(s) \quad \text{where } J(s) = \mathcal{L}[j(t)](s).$$

$$\mathcal{L}[e^{at} j(t)](s) = J(s-a) \quad \text{where } J(s) = \mathcal{L}[j(t)](s).$$

$$\mathcal{L}[u(t-c)j(t-c)](s) = e^{-cs} J(s) \quad \text{where } J(s) = \mathcal{L}[j(t)](s) \\ \text{and } u \text{ is the unit step function.}$$