Quiz 4 Solutions, Math 246, Professor David Levermore
Tuesday, 6 October 2015

(1) [2] What is the interval of definition for the solution to the initial-value problem

$$u''' + \frac{5}{1+t}\,u'' - \frac{e^t}{6-t}\,u = \frac{\cos(2t)}{7+t}\,, \qquad u(1) = u'(1) = u''(1) = -4\,.$$

Solution. This linear equation is in normal form. One of its coefficients is undefined at $t = -1$ and continuous elsewhere while the other is undefined at $t = 6$ and continuous elsewhere. Its forcing is undefined at $t = -7$ and continuous elsewhere. The initial time is $t = 1$. Therefore the interval of definition is $(-1, 6)$.

(2) [3] Compute the Wronskian $W[Z_1, Z_2](t)$ of the functions $Z_1(t) = 1-t$ and $Z_2(t) = e^t$. (Evaluate the determinant and simplify.)

Solution. Because $Z_1'(t) = -1$ and $Z_2'(t) = e^t$, the Wronskian is

$$\begin{aligned} W[Z_1, Z_2](t) &= \det\begin{pmatrix} Z_1(t) & Z_2(t) \\ Z_1'(t) & Z_2'(t) \end{pmatrix} = \det\begin{pmatrix} 1-t & e^t \\ -1 & e^t \end{pmatrix} \\ &= (1-t)e^t - (-1)e^t = (2-t)e^t\,. \end{aligned}$$

(3) [1] Suppose that $X_1(t)$, $X_2(t)$, and $X_3(t)$ are solutions of the differential equation

$$x''' + b(t)x' + c(t)x = 0\,,$$

where $b(t)$ and $c(t)$ are continuous over $(-9, 9)$. Suppose that $W[X_1, X_2, X_3](2) = 3$. What is $W[X_1, X_2, X_3](-4)$?

Solution. The equation is in normal form and has coefficients that are continuous over $(-9, 9)$. Because it is third-order while the coefficient of x'' is zero, the Abel Theorem implies that $W[X_1, X_2, X_3](t)$ is constant over $(-9, 9)$. We thereby conclude that $W[X_1, X_2, X_3](-4) = W[X_1, X_2, X_3](2) = 3$.

(4) [4] Given that e^{5t} and e^{-5t} are linearly independent solutions of $y'' - 25y = 0$, find the natural fundamental set of solutions of this equations associated with $t = 0$.

Solution. The general initial-value problem associated with $t = 0$ is

$$y'' - 25y = 0\,, \qquad y(0) = y_0\,, \quad y'(0) = y_1\,.$$

Because e^{5t} and e^{-5t} are linearly independent solutions, we see a general solution is $Y(t) = c_1 e^{5t} + c_2 e^{-5t}$. Then $Y'(t) = 5c_1 e^{5t} - 5c_2 e^{-5t}$ and the initial conditions yield

$$y_0 = Y(0) = c_1 + c_2\,, \qquad y_1 = Y'(0) = 5c_1 - 5c_2\,.$$

It follows that $5y_0 + y_1 = 10c_1$ and $5y_0 - y_1 = 10c_2$, whereby

$$c_1 = \tfrac{1}{2}y_0 + \tfrac{1}{10}y_1\,, \qquad c_2 = \tfrac{1}{2}y_0 - \tfrac{1}{10}y_1\,.$$

Hence, the solution of the general initial-value problem is

$$Y(t) = (\tfrac{1}{2}y_0 + \tfrac{1}{10}y_1)\,e^{5t} + (\tfrac{1}{2}y_0 - \tfrac{1}{10}y_1)\,e^{-5t} = \tfrac{1}{2}(e^{5t} + e^{-5t})y_0 + \tfrac{1}{10}(e^{5t} - e^{-5t})y_1\,.$$

Therefore the natural fundamental set of solutions associated with $t = 0$ is

$$N_0(t) = \tfrac{1}{2}(e^{5t} + e^{-5t})\,, \qquad N_1(t) = \tfrac{1}{10}(e^{5t} - e^{-5t})\,.$$