Quiz 5 Solutions, Math 246, Professor David Levermore
Tuesday, 13 October 2015

(1) [5] Give a general solution of the equation

$$(\mathrm{D}-5)^4(\mathrm{D}^2+6\mathrm{D}+13)^2 y = 0\,, \qquad \text{where} \quad \mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}t}\,.$$

Solution. This is an eighth-order, homogeneous, linear differential equation with constant coefficients. Its characteristic polynomial is

$$p(z) = (z-5)^4(z^2+6z+13)^2 = (z-5)^4\big((z+3)^2+2^2\big)^2\,,$$

which has roots 5, 5, 5, 5, $-3+i2$, $-3+i2$, $-3-i2$, and $-3-i2$. A general solution of the differential equation is

$$\begin{aligned} y(t) &= c_1 e^{5t} + c_2 t\, e^{5t} + c_3 t^2 e^{5t} + c_4 t^3 e^{5t} \\ &\quad + c_5 e^{-3t}\cos(2t) + c_6 e^{-3t}\sin(2t) + c_7 t\, e^{-3t}\cos(2t) + c_8 t\, e^{-3t}\sin(2t)\,. \end{aligned}$$

The reasoning is as follows.

- the fourfold real root 5 yields the solutions e^{5t}, $t\,e^{5t}$, $t^2 e^{5t}$, and $t^3 e^{5t}$;
- the double conjugate pair $-3 \pm i2$ yields the solutions

$$e^{-3t}\cos(2t)\,, \quad e^{-3t}\sin(2t)\,, \quad t\,e^{-3t}\cos(2t)\,, \quad \text{and} \quad t\,e^{-3t}\sin(2t)\,.$$

(2) [3] Give the degree, characteristic, and multiplicity for the forcing term of the equation

$$v'' + 6v' + 13v = 4t^5 e^{-3t}\cos(2t)\,.$$

Solution. This is a second-order, nonhomogeneous, linear equation with constant coefficients. Its characteristic polynomial is $p(z) = z^2+6z+13 = (z+3)^2+2^2$, which has roots $-3 \pm i2$.

The forcing term $4t^5 e^{-3t}\sin(2t)$ has degree $d = 5$, characteristic $\mu + i\nu = -3 + i2$, and multiplicity $m = 1$.

(3) [2] Give a particular solution of the equation

$$x'' + 6x' + 13x = 16e^{-t}\,.$$

Solution. This is a second-order, nonhomogeneous, linear equation with constant coefficients. Its characteristic polynomial is $p(z) = z^2 + 6z + 13 = (z+3)^2 + 2^2$, which has roots $-3 \pm i2$. Its forcing has degree $d = 0$, characteristic $\mu + i\nu = -1$, and multiplicity $m = 0$.

Key Identity Evaluations. Because $d + m = 0$, we need just the Key Identity

$$\mathrm{L}(e^{zt}) = (z^2 + 6z + 13)e^{zt}\,.$$

By setting $z = -1$ we find that

$$\mathrm{L}(e^{-t}) = (1 - 6 + 13)e^{-t} = 8e^{-t}\,.$$

After multiplying both sides of this equation by 2, we can read off that a particular solution is $X_P(t) = 2e^{-t}$.