

Quiz 6 Solutions, Math 246, Professor David Levermore
Tuesday, 20 October 2015

- (1) [3] Find the Green function $g(t)$ associated with the differential operator $D^2 + 6D + 9$.

Solution. The Green function $g(t)$ satisfies the initial-value problem

$$g'' + 6g' + 9g = 0, \quad g(0) = 0, \quad g'(0) = 1.$$

The characteristic polynomial is $p(z) = z^2 + 6z + 9 = (z + 3)^2$, which has the double real root -3 . Therefore $g(t)$ has the form

$$g(t) = c_1 e^{-3t} + c_2 t e^{-3t}.$$

Because $g'(t) = -3c_1 e^{-3t} + c_2(1 - 3t)e^{-3t}$, the initial conditions imply

$$0 = g(0) = c_1 e^0 + c_2 \cdot 0 e^0 = c_1,$$

$$1 = g'(0) = -3c_1 e^0 + c_2(1 - 3 \cdot 0)e^0 = -3c_1 + c_2.$$

The solution of this system is $c_1 = 0$, $c_2 = 1$. Therefore the Green function $g(t)$ is

$$g(t) = t e^{-3t}.$$

- (2) [3] Give a particular solution of the equation $x'' + 6x' + 9x = 10e^{-3t}$.

Solution. This is a second-order, nonhomogeneous, linear equation with constant coefficients. Its characteristic polynomial is $p(z) = z^2 + 6z + 9 = (z + 3)^2$, which has roots -3 and -3 . Its forcing has characteristic form with degree $d = 0$, characteristic $\mu + i\nu = -3$, and multiplicity $m = 2$. Let $L = D^2 + 6D + 9$.

Zero Degree Formula. Because $d = 0$, $\mu + i\nu = -3$, and $m = 2$, we may use the zero degree formula with $m = 2$. Because $p(z) = z^2 + 6z + 9$, we see that $p''(z) = 2$ and that

$$L\left(\frac{t^2 e^{-3t}}{p''(-3)}\right) = L\left(\frac{t^2 e^{-3t}}{2}\right) = e^{-3t}.$$

Therefore a particular solution of $L(x) = 10e^{-3t}$ is $x_P(t) = 5t^2 e^{-3t}$.

Remark. Had you forgotten the zero degree formula then you could have derived it by Key Identity Evaluations as in the following solution.

Key Identity Evaluation. Because $m = 2$ and $d + m = 2$, we need the second derivative of the Key Identity, which we compute as

$$L(e^{zt}) = (z^2 + 6z + 9) e^{zt},$$

$$L(t e^{zt}) = (z^2 + 6z + 9) t e^{zt} + (2z + 6) e^{zt},$$

$$L(t^2 e^{zt}) = (z^2 + 6z + 9) t^2 e^{zt} + 2(2z + 6) t e^{zt} + 2 e^{zt}.$$

By evaluating the second derivative at the forcing characteristic $z = -3$ we see that

$$\begin{aligned} L(t^2 e^{-3t}) &= ((-3)^2 + 6 \cdot (-3) + 9) t^2 e^{-3t} + 2 \cdot (2 \cdot (-3) + 6) t e^{-3t} + 2 e^{-3t} \\ &= 2e^{-3t}. \end{aligned}$$

Therefore a particular solution of $L(x) = 10e^{-3t}$ is $x_P(t) = 5t^2 e^{-3t}$.

Undetermined Coefficients. Because $d = 0$, $\mu + i\nu = -3$ and $m = 2$, there is a particular solution of $L(x) = 10e^{-3t}$ in the form

$$x_P(t) = At^2e^{-3t}.$$

Because

$$x'_P(t) = -3At^2e^{-3t} + 2At e^{-3t}, \quad x''_P(t) = 9At^2e^{-3t} - 12At e^{-3t} + 2Ae^{-3t},$$

we find that

$$\begin{aligned} x''_P + 6x'_P + 9x_P &= (9At^2e^{-3t} - 12At e^{-3t} + 2Ae^{-3t}) \\ &\quad + 6(-3At^2e^{-3t} + 2At e^{-3t}) + 9At^2e^{-3t} \\ &= 2Ae^{-3t}. \end{aligned}$$

By setting $2A = 10$ we find $A = 5$ and the particular solution $x_P(t) = 5t^2e^{-3t}$.

Green Function. The equation is already in normal form. To apply the Green method, we must first show (as in problem 1) that the Green function is given by $g(t) = te^{-3t}$. Then a particular solution of $L(x) = 10e^{-3t}$ is given by

$$\begin{aligned} x_p(t) &= \int_0^t g(t-s)f(s) ds = \int_0^t (t-s)e^{3s-3t} \cdot 10e^{-3s} ds = 10e^{-3t} \int_0^t (t-s) ds \\ &= 10e^{-3t}t \int_0^t ds - 10e^{-3t} \int_0^t s ds = 10e^{-3t}t^2 - 10e^{-3t}\frac{1}{2}t^2 = 5t^2e^{-3t}. \end{aligned}$$

This is the unique particular solution that satisfies $x_P(0) = 0$ and $x'_P(0) = 0$.

Remark. This is the same particular solution that was obtained by both Key Identity Evaluations and Undetermined Coefficients. In general the Green function method yields a different solution than those methods.

Remark. The Green Function method usually takes much longer than either the Key Identity Evaluations or Undetermined Coefficients methods! In the above problem the integrals are extremely simple, so the difference is much less dramatic than in most problems, but even in this case it takes longer because $g(t)$ has to be computed.

- (3) [4] A spring-mass system is governed by the initial-value problem

$$h'' + 2\mu h' + 9h = 0, \quad h(0) = 0, \quad h'(0) = 3.$$

- (a) Determine the natural frequency and period of the spring.
 (b) For what value of μ is the system critically damped?

Solution (a). Because the equation is in normal form, the natural frequency ω_o is

$$\omega_o = \sqrt{9} = 3.$$

The natural period is thereby $T_o = 2\pi/\omega_o = \frac{2\pi}{3}$.

Solution (b). The characteristic polynomial is

$$p(z) = z^2 + 2\mu z + 9 = (z + \mu)^2 + 9 - \mu^2.$$

The system will be critically damped when this polynomial has a double real root. This happens when $9 - \mu^2 = 0$, which is when

$$\mu = \sqrt{9} = 3.$$