Quiz 7 Solutions, Math 246, Professor David Levermore
Tuesday, 3 November 2015

Short Table: $\mathcal{L}[t^n](s) = \dfrac{n!}{s^{n+1}}$ for $s > 0$, $\mathcal{L}[u(t-c)j(t-c)](s) = e^{-cs}\mathcal{L}[j](s)$.

(1) [4] Use the definition of the Laplace transform to compute $\mathcal{L}[f](s)$ for the function $f(t) = e^{-2t}u(t-3)$, where u is the unit step function.

Solution. By the definitions of the Laplace transform and the unit step function

$$\begin{aligned}\mathcal{L}[f](s) &= \lim_{T\to\infty}\int_0^T e^{-st}f(t)\,\mathrm{d}t = \lim_{T\to\infty}\int_0^T e^{-st}e^{-2t}u(t-3)\,\mathrm{d}t \\ &= \lim_{T\to\infty}\int_3^T e^{-st}e^{-2t}\,\mathrm{d}t = \lim_{T\to\infty}\int_3^T e^{-(s+2)t}\,\mathrm{d}t\,.\end{aligned}$$

For $s \leq -2$ the above limit diverges because $e^{-(s+2)t} \geq 1$. For $s > -2$

$$\int_3^T e^{-(s+2)t}\,\mathrm{d}t = -\frac{e^{-(s+2)t}}{s+2}\bigg|_3^T = \frac{e^{-(s+2)3}}{s+2} - \frac{e^{-(s+2)T}}{s+2}\,,$$

whereby

$$\mathcal{L}[f](s) = \lim_{T\to\infty}\left[\frac{e^{-(s+2)3}}{s+2} - \frac{e^{-(s+2)T}}{s+2}\right] = \frac{e^{-(s+2)3}}{s+2} \quad \text{for } s > -2\,.$$

(2) [3] Find the Laplace transform $V(s)$ of the solution $v(t)$ of the initial-value problem

$v'' + 4v = 0$, $v(0) = 0$, $v'(0) = 2$. DO NOT solve for $v(t)$, just $V(s)$!

Solution. The Laplace transform of the initial-value problem gives

$$\mathcal{L}[v''](s) + 4\mathcal{L}[v](s) = 0\,,$$

where

$$\mathcal{L}[v](s) = V(s)\,, \qquad \mathcal{L}[v'](s) = sV(s) - v(0) = sV(s)\,,$$
$$\mathcal{L}[v''](s) = s\big(sV(s)\big) - v'(0) = s^2V(s) - 2\,.$$

Hence,

$$(s^2+4)V(s) - 2 = 0\,, \qquad \Longrightarrow \qquad V(s) = \frac{2}{s^2+4}\,.$$

(3) [3] Find the Laplace transform $\mathcal{L}[f](s)$ of the function

$$f(t) = \begin{cases} t^2 & \text{for } 0 \leq t < 2\,, \\ 4 & \text{for } 2 \leq t\,. \end{cases}$$

Solution. Because $f(t) = t^2 + u(t-2)\big(4 - t^2\big) = t^2 + u(t-2)\,j(t-2)$, where $j(t) = 4 - (t+2)^2 = -4t - t^2$, the short table above shows

$$\begin{aligned}\mathcal{L}[f](s) &= \mathcal{L}[t^2](s) + \mathcal{L}[u(t-2)\,j(t-2)](s) = \mathcal{L}[t^2](s) + e^{-2s}\mathcal{L}[j](s) \\ &= \mathcal{L}[t^2](s) - e^{-2s}\big(4\mathcal{L}[t](s) + \mathcal{L}[t^2](s)\big) = \frac{2}{s^3} - e^{-2s}\left(\frac{4}{s^2} + \frac{2}{s^3}\right), \quad \text{for } s > 0\,.\end{aligned}$$