Quiz 8 Solutions, Math 246, Professor David Levermore
Tuesday, 10 November 2015

(1) [3] Transform the equation $u''' + (u'')^2 - u^3 u' - t^2 e^u = e^t$ into a first-order system of ordinary differential equations.

Solution. Because the equation is third order, the first-order system must have dimension three. The simplest such first-order system is

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ e^t - x_3^2 + x_1^3 x_2 + t^2 e^{x_1} \end{pmatrix}, \quad \text{where} \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} u \\ u' \\ u'' \end{pmatrix}.$$

(2) [2] Compute $\begin{pmatrix} 1 & -2 \\ -2 & 6 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Solution.

$$\begin{pmatrix} 1 & -2 \\ -2 & 6 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \cdot 2 + (-2) \cdot 1 \\ (-2) \cdot 2 + 6 \cdot 1 \end{pmatrix} = \begin{pmatrix} 2 - 2 \\ -4 + 6 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$$

(3) [2] Compute $\begin{pmatrix} 11 & 4 \\ 2 & 1 \end{pmatrix}^{-1}$.

Solution.

$$\det\begin{pmatrix} 11 & 4 \\ 2 & 1 \end{pmatrix} = 11 \cdot 1 - 2 \cdot 4 = 11 - 8 = 3, \qquad \begin{pmatrix} 11 & 4 \\ 2 & 1 \end{pmatrix}^{-1} = \frac{1}{3}\begin{pmatrix} 1 & -4 \\ -2 & 11 \end{pmatrix}.$$

(4) [3] Give the interval of definition for the solution of the initial-value problem

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \dfrac{5}{9 - t^2} & -\dfrac{3}{t} \\ -\dfrac{3}{t} & \dfrac{5}{9 - t^2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \qquad \begin{pmatrix} x(-1) \\ y(-1) \end{pmatrix} = \begin{pmatrix} 7 \\ -2 \end{pmatrix}.$$

Solution. This first-order homogeneous linear system is already in normal form. The coefficients $5/(9 - t^2)$ are undefined at $t = -3$ and $t = 3$ and are continuous elsewhere. The coefficients $-3/t$ are undefined at $t = 0$ and are continuous elsewhere. The initial time is $t = -1$. Therefore the interval of definition for the solution of the initial-value problem is $(-3, 0)$ because

- all the coefficients are continuous over $(-3, 0)$,
- the initial time $t = -1$ lies within $(-3, 0)$,
- the coefficients $5/(9 - t^2)$ are undefined at $t = -3$,
- the coefficients $-3/t$ are undefined at $t = 0$.